Unwinding Infinitary Arguments in Combinatorics

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2. Proofs from ergodic Ramsey Theory,
3. Proofs using ultraproducts.

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Consider the following situation:

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There are two main approaches to this situation:

1. Invent a new, more explicit proof,
2. Use formal means to extract bounds from the non-effective proof.

Non-effective proofs often have intermediate stages for which *no bounds can possibly exist*.

A typical example is that the non-effective proof might use the convergence of some sequence

$$\forall \epsilon > 0 \exists N_\epsilon \forall n, m > N_\epsilon \ |s_n - s_m| < \epsilon.$$

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Note that the statement of convergence if Π^0_3: it has the syntactic form

$$\forall x \exists y \forall z \ \phi(x, y, z)$$

where ϕ itself is computable.

Given a Π^0_3 statement

$$\forall x \exists y \forall z \ \phi(x, y, z),$$

the functional interpretation gives us the following equivalent statement

$$\forall x \forall F \exists y_F \forall z \leq F(y_F) \ \phi(x, y_F, z).$$

For example, the *metastable convergence* of a sequence says

$$\forall \epsilon > 0 \forall F \exists N \forall n, m \in [N, F(N)] \ |s_n - s_m| < \epsilon.$$

A Π^1_2 sentence asserts that for any set X, there exists a set Y have some property

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Theorem (Infinite Pigeonhole Principle)

If $c : \mathbb{N} \to [0, r]$ then there is an infinite set Y such that $c \upharpoonright Y$ is constant.

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Theorem (Infinite Ramsey's Theorem)

If $c : [\mathbb{N}]^2 \to [0, r]$ then there is an infinite set Y such that $c \upharpoonright [Y]^2$ is constant.

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Theorem (Higman's Theorem)

If Q is a well-quasi-ordering then $Q^{<\omega}$ is a well-quasi-ordering.

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where ϕ itself is arithmetic (does not involve further quantifiers over sets).

Theorem (Menger's Theorem for Countable Graphs)

Let (G, E) be a countable graph and let $A, B \subseteq G$. Then there is a set S of pairwise disjoint A-B paths and a set P consisting of exactly one point from each element of S such that every path from A to B intersects an element of P.

Question (Reverse Mathematics)

What theory of second order arithmetic is $\forall X \exists Y \varphi(X, Y)$ equivalent to?

Question (Computability Theory)

Given a particular X, how complicated do the witnesses Y have to be?

Definition

The *leftmost path principle* states that if T is a tree of finite sequences and there is any infinite path through T then there is a leftmost infinite path through T (according to the lexicographic ordering).

This is a Π^1_3 sentence.

Theorem

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Theorem

1. *The leftmost path principle is equivalent to* $\mathbf{\Pi^1_1 - CA_0}$.
2. *(Marcone) The leftmost path principle implies the Generalized Higman's Theorem (and therefore the Nash-Williams Theorem).*
3. *(Shafer) The leftmost path principle implies Menger's Theorem for countable graphs.*
4. *No* Π^1_2 *sentence can be equivalent to the leftmost path principle.*

Question

Given a proof of a Π^1_2 sentence from the leftmost path principle, is there a systematic way of extracting a complexity bound on the witnesses?

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can be replaced by instances of the Γ-*relative leftmost path principle*:

> *For any ill-founded tree T of finite sequences and any operation Γ mapping sets to countable collections of sets, there exists a path Λ through T such that no path $\Lambda' \in \Gamma(\Lambda)$ is to the left of Λ*

Definition

$\mathbf{\Sigma}_0$-$\mathbf{LPP_0}$ is the theory consisting of $\mathbf{RCA_0}$ together with the Σ_0-relatively leftmost path principle where $\Sigma_0(\Lambda)$ is the collection of paths computable in Λ.

Theorem

$\mathbf{\Sigma}_0$-$\mathbf{LPP_0}$ *implies* $\mathbf{ATR_0}$.

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Both Menger's Theorem and the Generalized Higman's Theorem appear to use the following principle:

Definition

$\mathbf{TLPP_0}$ is the theory consisting of $\mathbf{RCA_0}$ together with the principle that whenever α is an ordinal, the (α)-relatively leftmost path principle holds (where $(\alpha)(\Lambda)$ is the collection of paths computable $\Lambda^{(\alpha)}$).

In particular, note that $\mathbf{TLPP_0}$ is given by a Π^1_2 axiom.