AN INFINITE SELF DUAL RAMSEY THEOEREM

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Definitions

Let K and L be finite linear orders. By an rigid surjection $t : L \to K$ we mean a surjection with the additional property that images of initial segments of the domain are also initial segments of the range.We call a pair (t, i) a connection between K and L if $t : L \to K$, $i : K \to L$ such that for all $x \in L$:

$$t(i(x)) = x \text{ and } \forall y \leq i(x) \Rightarrow t(y) \leq x.$$

It is easy to see that if (t, i) is a connection then t is a rigid surjection and i is an increasing injection. Similarly we define (s, j) a connection between ω and K, if $s : \omega \to K, j : K \to \omega$ such that for all $x \in L$:

$$s(j(x)) = x \text{ and } \forall y \leq j(x) \Rightarrow s(y) \leq x.$$

Once more if (s, j) is a connection then s is a rigid surjection and j is an increasing injection.

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$F^A_{\omega,\omega} = \{ (r,c) : r : \omega \to \omega \cup A,\ c : \omega \to \omega, c$ is an increasing injection: $r(c(x)) = x$ and $y \leq c(x) \Rightarrow r(y) \leq x\}$

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$F^A_{\omega,K} = \{\,(s,j) :\ s : \omega \to K \cup A, j : K \to \omega : j$ is an increasing injection such that $s(j(x)) = x, y \leq j(x) \Rightarrow s(y) \leq x\,\}$.

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For $k \in \omega$,
$(r, c)^K_A = \{\, (s, j) \in F_{\omega,K} : (s, j) \leq (r', c'), (r', c') \in (r, c)^A_\omega \}$.

Let now $(r,c)^{\star}_{A} = \{\,(t,\varnothing) : (t,\varnothing) \preceq (r',c'), (r',c') \in (r,c)^{\omega}_{A}$ and if the length of t is M, then $r'(M) = 0, t \restriction M \subseteq A\,\}$. By $\varnothing$ emphasize that the increasing injections in the second coordinate do not have A in their domain.

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$[r,c]^{L}_A = \{ (t,i) : (t,i) \preceq (r',c')$ where $(r',c') \in (r,c)^{\omega}_A$, the domain of i is equal to L and $r'(lh(t,i)) = L \}$.

Let $(t,i) \preceq (r,c), (t,i) \in [r,c]^L_A$, where its length is M and the domain of i is equal to L i.e. $(t,i) \in F^A_{M,L}$, by $(t^\star, i^\star) \in (r,c)^{L+1}_A$ we mean the unique predecessor of (r,c) on which $i^\star$ has domain equal to $L+1$, $i^\star \restriction L = i \restriction L$, $t^\star \restriction M = t \restriction M \subseteq \{0, \ldots L-1\}$ $t^\star(M) = L$ and $r(lh(t^\star, i^\star)) = L+1$.

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$(s,j) \cdot (r,c) = (s \circ r, c \circ j)$ so the order of composition in the two coordinates is not the same.

Ramsey Theorem

Let l, K be natural numbers. For any l-coloring of all increasing injections $j : K \to \omega$ there exists an increasing injection $j_0 : \omega \to \omega$ such that the set $\{ j_0 \circ j : j : K \to \omega \}$ is monochromatic.

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Graham-Rothschild

Let l, K, L, M be natural numbers. For any l-coloring of all rigid surjections $s : K \to L$ there exists a rigid surjection $s_0 : K \to M$ such that the set $\{\, t \circ s_0 : t : M \to L$ a rigid surjecton $\,\}$ is monochromatic

Carlson-Simpson

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Solecki

For any finite coloring of $F_{K,L}$, there exists $(s_0, j_0) \in F_{K,M}$ such that the set $\{ (t, i) \circ (s_0, j_0) : (t, i) \in F_{M,L} \}$ is monochromatic.

MAIN THEOREM

THEOREM

Let $l > 0$ be a natural number. Let K be a finite linear order. For each $l-$coloring of all connections between ω and K, that is Borel, there exists a connection $(r_0, c_0) : \omega \leftrightarrow \omega$ such that the set $\{ (s,j) \cdot (r_0, c_0) : (s,j) : \omega \leftrightarrow K \}$ is monochromatic.

Theorem 1

If $F^A_{\omega,K} = C_0, \cup \ldots, \cup C_{l-1}$ where each C_i is Borel, then there exists $(r_0, c_0) \in F^A_{\omega,\omega}$ such that $(r_0, c_0)^K_A \subseteq C_k$ for some $k \in l$.

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proof

By induction on K

Lemma 1, K=0

If $(r, c) \in F^A_{\omega,\omega} and (r, c)^0_A = C_0 \cup \cdots \cup C_{l-1}$ where each C_k is Borel, then there exists $(r', c') \in (r, c)^\omega_A$ such that $(r', c')^0_A \subseteq C_k$ for some $k \in l$.

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proof

Note that the coloring does not depend on the second coordinate so in particular this theorem reduces to the Carlson-Simpson theorem.

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For an infinite sequence $X = (x_n)_{n\in\omega}$ of elements of W_{Lv}, by $[X]_A$ we denote the partial semigroup of W_A generated by X as follows:
$[X]_A = \{ x_{n_0}(\alpha_0) \frown \cdots \frown x_{n_k}(\alpha_k) : n_0 < \cdots < n_k, \alpha_i \in A (i \leq k) \}$.

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LVWT, (Todorcevic)

Let A be a finite alphabet, then for any finite coloring of W_A there is an infinite sequence $X = (x_n)_{n \in \omega}$ of left variable-words and a variable free word w_0 such that the translate $w_0 \frown [X]_A$ of the partial semigroup of W_A generated by X is monochromatic.

Lemma 2

If $(r, c) \in F^{A}_{\omega,\omega}$ and $(r, c)^{\star}_{A} = C_0 \cup \cdots \cup C_{l-1}$, then there exists $(r', c') \in (r, c)^{\omega}_{A}$ such that $(r', c')^{\star}_{A} \subseteq C_k$ for some k.

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proof

Code each element of $(r, c)^\star_A$ by a word in W_A and color W_A accordingly.

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Consider the infinite word $w_0 \frown x_0 \frown \cdots \frown x_n \frown \ldots$. If at the i-th and j-th positions of the above infinite word there is a variable, where $i, j \in [lh(w_0 \frown \cdots \frown x_n), lh(w_0 \frown \cdots \frown x_{n+1})]$, then identify them in the equivalence relation. If in the i-th position there is a letter $\alpha \in A$ then $X_{r'}(i) = \alpha$. The resulting equivalence relation is such that $(X_{r'})^\star_A \subseteq C_k$ for fixed k.

Induction step

Assuming theorem 2 holds for $F^{A+1}_{\omega,K}$, where $A+1$ denotes a finite alphabet of cardinality $|A|+1$. Then Main Theorem holds for $F^{A}_{\omega,K+1}$.

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Let $F^{A}_{\omega,K+1} = C_0 \cup \cdots \cup C_{l-1}$ be a Borel coloring. There exists a canonical homeomorphism between $[(t^\star, i^\star), (r', c')]^{K+1}_{A}$ and $F^{A+1}_{\omega,K}$.

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Then the last Lemma implies that we get an $(r', c') \in (r, c)^\omega_A$ such that $(r', c')^\star_A \subseteq C'_h$ for some fixed h' and therefore (r', c') is the desired one.

Theorem

$\langle F_{\omega,\omega}, u, \leq \rangle$, where $u : F_{\omega,\omega} \times \omega \to \mathcal{A}F_{\omega,\omega} = \bigcup_{N \leq M} F_{M,N}$ by $u_N((r,c)) = (t,i), (t,i) \in F_{M,N}, (t,i) \preceq (r,c)$.
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Theorem

Let $F_{\omega,\omega} = C_0 \cup \cdots \cup C_{l-1}$ be a Baire or Suslin measurable coloring. There exists $(r,c) \in F_{\omega,\omega}$ such that $(r,c)^{\omega}$ is monochromatic.

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Theorem

$c : F_{\omega,K} \to h$ is a finite coloring that is Baire measurable relative to the topology defined just above. Then there exists an $(r,c) \in F_{\omega,\omega}$ such that the family $F_{\omega,K}|(r,c) = \{ (s,j) \in F_{\omega,K} : (s,j) \leq (r,c) \}$ is c-monochromatic.

Conclusion

Our Infinite Self Dual Ramsey Theorems hold in the realm of Baire measurable colorings.