# CANONICAL RAMSEY THEORY ON POLISH SPACES

Jindřich Zapletal
Academy of Sciences, Czech Republic
University of Florida

with Vladimir Kanovei and Marcin Sabok

**Ramsey theorem, countable:**

For every partition of $[\omega]^2$ into two parts, there is a homogeneous set.

**Ramsey theorem, Polish:**

For every partition of $[\omega]^{\aleph_0}$ into two Borel parts, there is a homogeneous set.

**Canonization theorem, countable:**

For every equivalence $E$ on $[\omega]^2$ there is a set on which it is simple.

**Canonization theorem, Polish:**

For every Borel equivalence $E$ on a Polish space, there is a perfect set on which it is simple.

## Main problem.

**Given** a Polish space $X$, a $\sigma$-ideal $I$ on it, and a Borel $E$-equivalence relation on it, **is there** a Borel $I$-positive set on which $B$ is simple?

## Possible answers.

- there is a Borel positive set on which $E$ is one of several prescribed forms;

- there is a Borel positive set $B$ such that $E \restriction B$ is strictly simpler than $E$ in the sense of Borel reducibility;

- on every Borel positive set $E$ keeps the original complexity in the sense of Borel reducibility: $E$ is reducible to $E \restriction B$.

## Connections.

- Borel reducibility theory;
- forcing;
- mathematical analysis;
- finite Ramsey theory.

## Sample theorems I.

For every Borel equivalence relation $E$ on the unit circle $T$,

- either $T$ decomposes into countably many $E$-equivalence classes and a set of extended uniqueness;
- or there is a compact set of restricted multiplicity consisting of $E$-inequivalent elements.

## Sample theorems II.

For every equivalence relation $E$ on $[0,1]^\omega$ classifiable by countable structures, there is a product of perfect sets on which $E$ is smooth.

The equivalence relation $xE_1y$ if $x(n) = y(n)$ for all but finitely many $n$ keeps its complexity on every product of perfect sets.

## Sample theorems III.

For every Borel equivalence relation $E$ on $\omega^\omega$ classifiable by countable structures,

- either $\omega^\omega$ decomposes into countably many $E$-equivalence classes and a nondominating set;

- or there is a closed dominating set consisting of pairwise $E$-inequivalent elements.

There is a Borel equivalence relation which keeps its complexity on every Borel dominating set.

**Sample challenge.**

Find a countable equivalence relation $E$ and a $\sigma$-ideal $I$ such that $E$ is in the spectrum of $E$ while $E_0$ is not.