Encodable by thin sets

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$RT^n_{<\infty,l}$-encodable

- Let c be a coloring of all finite sets of size n (all subsets of ω) by finitely many colors, not necessarily computable.
- A set T is *l-thin* iff c uses at most l colors to color all the sets of size n from T and T is infinite. So $|c([T]^n)| \leq l$.
- A set S is *$RT^n_{<\infty,l}$-encodable* iff there is a coloring c (as above) such that every l-thin set T computes S, i.e. $S \leq_T T$.

Question

What sets are $RT^n_{<\infty,l}$-encodable? $RT^2_{2,1}$? $RT^1_{2,1}$? $RT^3_{5,4}$? $RT^4_{14,13}$?

$RT^n_{<\infty,l}$-encodable sets are always hyperarithmetic.

- Assume c witness that S is $RT^n_{<\infty,l}$-encodable.
- Given X there is an infinite thin set H for c such that $H \subseteq X$.
- A set S is *computably encodable* if for every infinite set X, there is an infinite subset H of X such that H computes S.
- By theorems of Jockusch and Soare and Solovay, the computably encodable sets are exactly the hyperarithmetic sets.

The $RT^2_{<\infty,1}$-encodable sets includes all hyperarithmetic sets

- The 1-thin sets are exactly the homogenous sets.
- (Solovay) S is hyperarithmetical iff S has a *modulus*, i.e. a function g such that, for all functions h, if $g \leq h$ then $S \leq_T h$.
- The interval $[x, y]$ is g-large iff $g(x) < y$.
- $c(x, y) = 1$ iff $[x, y]$ is g-large. (An unbalanced coloring.)
- Let H be a homogenous set for c. Fix $x \in H$. Then, for almost all $y \in H$, $[x, y]$ is g-large. So, for all $y \in H$, $[x, y]$ is g-large.
- Hence $g \leq_T p_H$.

For every hyperarithmetical set S there is a coloring (of the same Turing degree as S) such that every homogenous set computes S.

The $RT^n_{<\infty,l}$-encodable sets, for $l < 2^{n-1}$

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

For $l < 2^{n-1}$, the $RT^n_{<\infty,l}$-encodable sets are exactly the hyperarithmetic sets.

Again code in a modulus into all thin sets of a coloring.

For $2^{n-1} \leq l$, the coding does not work. In particular, for $n = 3$ and $l = 4$, the coding does not work.

$RT^1_{2,1}$-encodable

Theorem (Dzhafarov and Jockusch)

Only the computable sets are $RT^1_{2,1}$-encodable.

Let $c : \omega \to 2$. Let $A = c^{-1}(1)$.

Theorem (Dzhafarov and Jockusch)

Given A and a noncomputable X. There an infinite G such that $X \not\leq_T G$ and either $G \subseteq A$ or $G \subseteq \overline{A}$.

We will work forward the proof of this theorem over next few slides.

Definition (Strong Cone Avoidance)

Given a problem P and a noncomputable set X, there is a solution S that $X \not\leq_T S$.

Corollary

$RT^1_{2,1}$ satisfies strong cone avoidance. $RT^2_{2,1}$ does not.

WKL

Theorem

Let $T \subseteq 2^{<\omega}$ be infinite tree. Then T has an infinite path.

Lemma

A tree is finite iff there is an l such that for all $\sigma \in 2^l$, $\sigma \notin T$. This is Σ_1^T or c.e. in T. Moreover this is uniform.

Given X the characteristic function of X is a tree with a single path X. So *WKL* does not satisfy strong cone avoidance.

Cone Avoidance

Definition (Cone Avoidance)

Given a problem P and a set $X \not\leq_T P$, there is a solution S that $X \not\leq_T S$.

Theorem (Relativized to I)

WKL satisfy cone avoidance. I.e. for all infinite trees $T \leq_T I$ and all $X \not\leq_T I$, there is a path Z such that $X \not\leq_T Z \oplus I$.

Forcing – Infinite trees and generic paths

The forcing conditions are the infinite trees $\tilde{T}$ such that $\tilde{T} \subseteq T$ and $\tilde{T} \leq_T I$. The forcing extension is inclusion. A set of conditions is dense if every condition can be extended into the dense set. A object G (here a tree) is sufficiently generic if it is the limit of conditions meeting enough dense sets.

Lemma

For all k, the set of subtrees such that almost all nodes in the subtree extend some finite segment of size $n \geq k$ is dense. So a generic tree is a path though T.

Cone Avoidance

Let Φ be a Turing functional. Enough to extend a condition $\tilde{T}$ to force $\Phi^{G \oplus I} \neq X$.

- **Non commitment**: For some n, the subtree $\{\sigma \in \tilde{T} \mid \Phi^{\sigma \oplus I}(n)\uparrow\}$ is infinite. Then, by finite use principle, $\Phi^{G \oplus I}(n)\uparrow$. This tree is computable in I.
- **Commitment:** There is a n and $\sigma \in \tilde{T}$ such that $\Phi^{\sigma \oplus I}(n) \neq X(n)$ and the subtree $\{\tau \in \tilde{T} \mid \tau \preceq \sigma \text{ or } \sigma \preceq \tau\}$ is infinite.
- **Otherwise**. Then, for all n, there is an l such that, for all $\sigma \in \tilde{T} \cap 2^l$, $\Phi^{\sigma \oplus I}(n)\downarrow = X(n)$. Therefore $X \leq_T I$. Contradiction.

Back to $RT^1_{2,1}$

Theorem

Given A and noncomputable X. There a G such that either $X \not\leq_T G \cap A$ and $G \cap A$ is infinite or $X \not\leq_T G \cap \overline{A}$ and $G \cap \overline{A}$ is infinite.

Use conditions (F, I) where F is finite, I is infinite, $\max F < \min I$, and $X \not\leq_T I$. $(\tilde{F}, \tilde{I})$ extends (F, I) if $F \subseteq \tilde{F} \subseteq F \cup I$ and $\tilde{I} \subseteq I$. WLOG $I \cap A$ and $I \cap \overline{A}$ are both infinite. With enough genericity, both $G \cap A$ and $G \cap \overline{A}$ are infinite.

Strong Cone Avoidance of $RT^1_{2,1}$

Must extend (F, I) to show either $\Phi^{G\cap A} \neq X$ or $\Psi^{G\cap\overline{A}} \neq X$.

Definition

Let $P_{n,k}$ be the tree of $Z \subseteq I$ such that there is no $E \subseteq Z$ with $\Phi^{(F\cap A)\cup E}(n)\downarrow = k$ and no $E \subseteq (I - Z)$ with $\Psi^{(F\cap\overline{A})\cup E}(n)\downarrow = k$.

Lemma

These trees are uniformly computable in (F, I). So $P_{n,k} \leq_T I$.

Commitment: For some n and $k \neq X(n)$, $P_{n,k}$ is finite. So $I \cap A \notin P_{n,k}$. Extend.

Non commitment

Let $S = \{(n, k) \mid P_{n,k} \text{ is finite}\}$. S is c.e. in (F, I). For all n, $(n, 1 - X(n))$ not in S. If, for all n, $(n, X(n))$ in S, then X is computable from (F, I) or just I. Since $X \not\leq_T I$, there must be an $(n, X(n))$ not in S.

Fix such an n. $P_{n,X(n)}$ is infinite. Use **cone avoidance of WKL** to find a $Z \in P_{n,X(n)}$ such that $X \not\leq_T Z \oplus I$. If Z is infinite extend to (F, Z). Otherwise use $(F, I - Z)$.

This is called *thinning the reservoir*.

The $RT^n_{<\infty,l}$-encodable sets, for large l

Theorem (Wang)

For big l (in terms of n), the $RT^n_{<\infty,l}$-encodable sets are exactly the computable sets. For $n = 2, l = 2$ and, for $n = 3, l = 5$.

Use the *strong cone avoiding* of $RT^n_{<\infty,l}$, for big l.

This is an inductive forcing proof and relies on (strong) cone avoiding of earlier and other principles, like *COH*, *WKL*, $RT^1_{<\infty,1}$, $RT^2_{3,1}$, etc. Use Mathias like conditions like we used above.

A recap for $n = 1, 2, 3$

Only the computable sets are $RT^1_{<\infty,1}$-encodable. Same for $RT^2_{<\infty,2}$-encodable and $RT^3_{<\infty,5}$-encodable.

The hyperarithmetic sets are $RT^2_{<\infty,1}$-encodable and same for $RT^3_{<\infty,3}$-encodable.

What about $RT^3_{5,4}$?

Strong Nonarithmetical Cone Avoidance

Definition (Strong Nonarithmetical Cone Avoidance)

Given a problem P and a set X not arithmetical, there is a solution S that $X \not\leq_T S$.

Theorem

$RT^3_{5,4}$ satisfies strong nonarithmetical cone avoidance. So does $RT^n_{<\infty,2^{n-1}}$.

Very carefully choose the reservoir.

$RT^3_{5,4}$-encodable

Corollary

At best only the arithmetic sets are $RT^3_{5,4}$-encodable.

Another coding is needed to code the arithmetic sets. By necessity this coding will also provide a counterexample to strong cone avoiding.

Theorem

There is a Δ^0_2 coloring $c : [\omega]^3 \to 5$ such that every 4-thin set for c computes $0'$.

Our first attempt

Let g be a modulus of $0'$.

- Recall $[a, b]$ is *g-large* iff $g(a) \leq b$. Otherwise it is *g-small*.
- Let $i(x, y) = 1$ if $[x, y]$ is g-large and 0 otherwise.
- Let $c(x, y, z) = \langle i(x, y), i(y, z), i(x, z) \rangle$. This is a 5 coloring, some colors are missing.
- Apply $RT^3_{5,4}$ to c to get a thin set T.
- If any color but $\langle 0, 0, 1 \rangle$ is missed, T or a reduction of T has all g-large intervals and hence computes $0'$. The principal function dominates the modulus for $0'$.
- Need to learn more about missing the color $\langle 0, 0, 1 \rangle$.

GAP

A set H is *g-transitive* iff, for all $x < y < z$ in H, if $[x, y]$ and $[y, z]$ are g-small so is $[x, z]$. GAP is the statement that, for all g, an infinite g-transitive set exists. So the existence of a 4-thin set (when colored as above) without the color $\langle 0, 0, 1 \rangle$.

Theorem

GAP satisfies strong cone avoidance. Hence the above coloring does not show the arithmetic sets are $RT^3_{5,4}$-encodable. Also GAP follows from $RT^2_{2,1}$.

Perhaps should known this coding would fail since all hyperarithmetic sets have a modulus not just arithmetic sets. Needed to use some fact about non hyperarithmetic arithmetic sets.

Back to $RT^3_{5,4}$ and coding left c.e. increasing functions

Refine the above coloring c. We need to make it harder to avoid the color $\langle 0,0,1\rangle$. So we have to color more triples with color $\langle 0,0,1\rangle$ and less with color $\langle 0,0,0\rangle$.

The modulus g of $0'$ is a left c.e. increasing function with approximations $g_0, g_1, \ldots$ (the approximations are increasing).

- Define $j(x,y,z)$ is 1 iff $[x,z]$ is g-large or $[x,y]$ is g_z-large.
- Let $c(x,y,z) = \langle i(x,y), i(y,z), j(x,y,z)\rangle$. A 5 coloring.
- Apply $RT^3_{5,4}$ to c to get a thin set T.
- For all possible missed colors, a reduction of T has all g-large intervals and hence computes $0'$.

Theorem

c is Δ^0_2 and every 4-thin set for c computes $0'$.

Bounds for $n > 3$, Part I

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

For $l < 2^{n-1}$, the $RT^n_{<\infty,l}$-encodable sets are exactly the hyperarithmetic sets.

Theorem

$RT^n_{<\infty,2^{n-1}}$ satisfies strong nonarithmetical cone avoidance.

Theorem (Wang)

For big l (in terms of n), $RT^n_{<\infty,l}$ satisfies strong cone avoidance.

What is the definition of "big"?

Bounds for $n > 3$, Part II

Definition

Let $l_1 = 1$ and
$l_{n+1} = l_n + \sum_{i \in \{1,\dots n-1\}} l_{n-i} \cdot l_i \ + \sum_{i<j \in \{1,\dots n-1\}} l_{n-j} \cdot l_i \cdot 2^{j-i-1}$..

Theorem

$RT^n_{<\infty, l_{n+1}}$ *satisfies strong cone avoidance.*

Definition

Let $d_0 = 1$ and $d_{n+1} = d_n + \sum_{i=1}^{n} d_{n-i} \cdot 2^{i-1}$.

Theorem

For all n, there is a Δ^0_2 *coloring of* $[\omega]^n$ *such that every* $d_n - 1$*-thin set computes* $0'$.

Remaining Questions

- $l_5 = 14 > d_4 = 13$.
- What sets are $RT^4_{14,13}$ encodable? At best the arithmetic sets. At worst the computable sets.
- Is there a Δ^0_2 coloring of $[\omega]^4$ such that every 13-thin set computes $0'$?
- For $n > 4$, $l_{n+1} > d_n$.

Computable Coding via a Modulus

Question

How necessary is it to code via the use of a modulus?

Theorem

Fix a function g. Let f be a computable instance of $RT^n_{k+1,k}$ such that every infinite k-thin set computes a function dominating g. Then for every infinite f-thin set H, p_H is a modulus for g.

Our coding examples are not computable but arithmetic. Using the Limit Lemma they can be reflected into computable colorings with the same properties.

Arbitrary Coding via a Modulus

Theorem (Liu and Patey)

Fix a function g. Let f be an instance of $RT^n_{k+1,k}$ *such that every infinite k-thin set computes a function dominating g via some fixed Turing functional. Then for every infinite f-thin set H,* p_H *is a uniform modulus for g.*

For $l = 1$, our coding examples for $RT^n_{<\infty,l}$ are uniform. But for larger l they are uniform in the missed color. Can hypothesis of the above theorem be weaken to reflect this uniformity?

Lemma (Liu and Patey)

There is an instance for $RT^2_{2,1}$ *all of those homogenous sets compute* $0'$ *but by not computing a function dominating a modulus for* $0'$*.*