VC-dimension in groups

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19 July 2018
Ramsey Theory in Logic,
Combinatorics, and Complexity
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- $\Omega = \mathbb{R}^2$ and $\mathcal{S}$ is the collection of convex sets. $\mathrm{VC}(\mathcal{S}) = \infty$

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Corollary: If $(V, W; E)$ omits some bipartite graph $(V_0, W_0; E_0)$, with $|V_0|, |W_0| \leq k$, then $\mathrm{VC}(\mathcal{S}) \leq k + \log k$.

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Theorem (AIM group[1] 2017; Chernikov-Simon 2015)

Let G be a group and fix $A \subseteq G$ such that $\mathrm{VC}(A) < \infty$. If there is an invariant measure μ on G such that $\mu(A) > 0$, then A is piecewise syndetic.

[1] Chernikov, C., Freitag, Goldbring, Wagner

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Corollary (Density Hindman's Theorem for VC-sets)

Let G be an infinite torsion-free group. Fix $A \subseteq G$ and suppose there is an invariant measure μ on G with $\mu(A) > 0$. If $\mathrm{VC}(A) < \infty$ then there is $g \in G$ such that gA is an IP-set.

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(2) If G is pseudofinite then the Boolean algebra of internal sets is amenable.

Regularity

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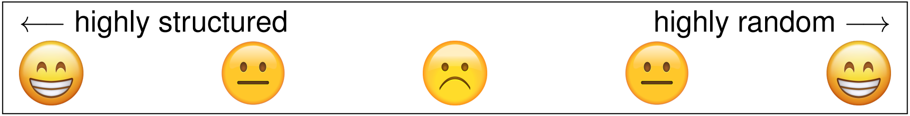

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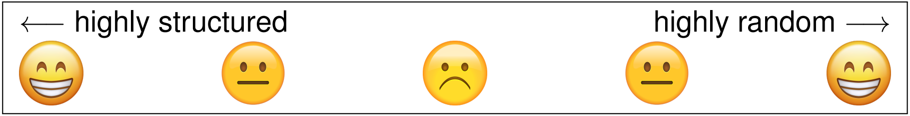

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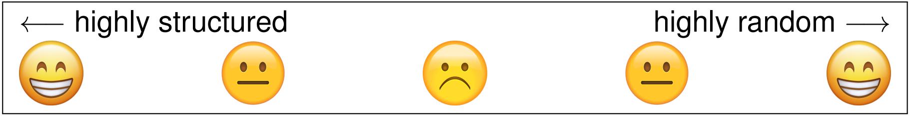

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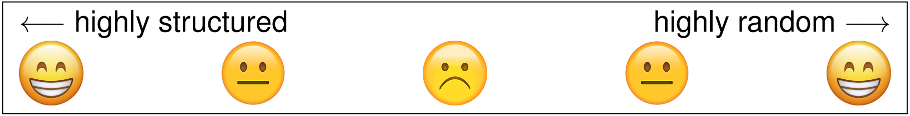

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- random: edges between pieces of regular pairs
- small: edges in some piece, or between pieces of irregular pairs

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- Chow, Lindqvist, Prendiville: Rado's criterion for partition regularity of sums of sufficiently many k^{th} powers.

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- C., Pillay, Terry (2017) “stable” subsets of finite groups

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Clause (*ii*) gives a regular partition for the bipartite graph on (G, G) induced by $xy \in A$, in which the pieces are the cosets of H and the regular pairs have density within ϵ of 0 or 1.

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There is a well-defined group operation on $\prod_{\mathcal{U}} G_s$, namely:

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Proposition: The Boolean algebra of internal sets is amenable. In particular, given $X = \prod_{\mathcal{U}} X_s$, define

$$\mu(X) = \lim_{\mathcal{U}} |X_s|/|G_s|.$$

Internal subsets of pseudofinite groups

Suppose $G = \prod_{\mathcal{U}} G_s$, as in the last slide.

Given a sequence $(X_s)_{s \in \mathbb{N}}$, with $X_s \subseteq G_s$, set $\prod_{\mathcal{U}} X_s := (\prod_{s \in \mathbb{N}} X_s)/\!\sim$.

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Łoś's Theorem: First-order properties of internal subsets of G correspond to $\mathcal{U}$-asymptotic properties of subsets of G_s.

VC-sets in pseudofinite groups

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- (iii) *Any internal set containing G_A^{00} has positive measure.*

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Suppose G has finite exponent. Then for any $\epsilon > 0$, there are:

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- From this, it follows that G_A^{00} is an intersection of *normal finite-index subgroups* of G in $\mathcal{B}(A)$.

Arithmetic regularity with bounded exponent

Theorem (CPT)

Fix $r, d \geq 1$. Suppose G is a finite group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

* *a normal subgroup $H \leq G$, of index $O_{r,d,\epsilon}(1)$,*
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If $G = \prod_{\mathcal{U}} G_s$ and $A = \prod_{\mathcal{U}} A_s$ then G has exponent r and $\mathrm{VC}(A) \leq d$.

Assuming $\mathcal{U}$ is nonprincipal, this contradicts the previous lemma.

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Recall: If $G = (\mathbb{Z}/p\mathbb{Z}, +)$ and $A = \{0, 1, \ldots, \frac{p-1}{2}\}$ then $\mathrm{VC}(A) \leq 3$.

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But we cannot have $|A \triangle D| < \frac{1}{2}|G|$, where D is a union of cosets of a subgroup of $\mathbb{Z}/p\mathbb{Z}$ whose index is independent of p.

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Setting: G is pseudofinite, and $A \subseteq G$ is internal with $\mathrm{VC}(A) < \infty$.

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So Γ^0 is an inverse limit of real tori (i.e. $\mathbb{T}^n$ for varying n).

Bohr sets

Definition

Given a group H, a homomorphism $\tau\colon H \to \mathbb{T}^n$, and some $\delta > 0$, set

$$B^n_{\delta,\tau} := \{x \in H : d(\tau(x), 0) < \delta\},$$

where d denotes the usual metric on $\mathbb{T}^n$, and 0 is the identity in $\mathbb{T}^n$.

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A quantitative version for finite abelian groups was obtained independently by Sisask (2018).

VC sets in nonabelian finite simple groups

Remark: If G is a nonabelian simple group, and $B \subseteq G$ is a Bohr set, then $B = G$.

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For any $d \geq 1$ and $\epsilon > 0$, there is $n = n(d, \epsilon)$ such that if G is a nonabelian finite simple group of size at least n, and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$, then $|A| < \epsilon|G|$ or $|A| > (1 - \epsilon)|G|$.

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By adapting the methods of Alon, Fox, and Zhao, and applying work of Gowers on “quasirandom groups”, one can give a direct proof of the previous corollary, which yields $\log(n(d, \epsilon)) \leq O((90/\epsilon)^{6d})$.

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If one also uses the classification of finite simple groups then 6 can be improved to $3 + \mu$ for any fixed $\mu > 0$.

Bounds in the bounded exponent case

The following result is proved by combining methods of Alon, Fox, and Zhao with structural results on "approximate groups" due to Breuillard, Green, and Tao.

Theorem (C. 2018)

Fix $r, d \geq 1$. Suppose G is a finite group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

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* *a set $D \subseteq G$, which is a union of cosets of H, and*
* *a set $Z \subseteq G$, which is a union of cosets of H, with $|Z| < \epsilon|G|$,*

satisfying the following properties.

(i) $|A \triangle D| < \epsilon^4 |G|$.

(ii) *For any $g \notin Z$, either $|gH \cap A| < \epsilon|H|$ or $|gH \backslash A| < \epsilon|H|$.*

thank you