The Strength and Effective Content of Hindman's Theorem for Bounded Sums

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Collaborators and References

This is joint work with Barbara Csima, Damir Dzhafarov, Denis Hirschfeldt, Reed Solomon, and Linda Brown Westrick.

The first paper is “Effectiveness of Hindman’s Theorem for bounded sums", by Dzhfarov, J., Solomon, and Westrick in the Rodfest volume *Computability and Complexity*, published by Springer in 2017.

A second paper “The reverse mathematics of Hindman’s theorem for sums of exactly two elements" by the above authors together with Csima and Hirschfeldt will appear in *Computability*.

The second paper makes crucial use of an effective version of the Lovász Local Lemma due to Andrei Rumyantsev and Alexander Shen in “Probabilistic constructions of computable objects and a computable version of Lovász Local Lemma", Fundamenta Informaticae 132 (2014), no. 1, 1–14.

Hindman's Theorem

Theorem (Neil Hindman, 1974)

If the positive integers are colored with finitely many colors, there is an infinite set X of positive integers such that all finite nonempty sums of distinct elements of X have the same color.

This result will be denoted HT throughout this talk.

Notation

Let $\mathbb{N} = \{1, 2, \ldots\}$.

For $k \in \mathbb{N}$, identify k and $\{0, 1, \ldots, k-1\}$.

A *k-coloring* c of $\mathbb{N}$ is a function $c : \mathbb{N} \to k$.

A set $H \subseteq \mathbb{N}$ is *monochromatic* for a k-coloring c of $\mathbb{N}$ if c is constant on H.

For $X \subseteq \mathbb{N}$, $\mathrm{FS}(X) = \{\Sigma(F) : F \subseteq X \ \& \ 0 < |F| < \infty\}$, where $\Sigma(F)$ is the sum of all the elements of F.

Thus HT asserts that for every $k \in \mathbb{N}$ and every k-coloring c of $\mathbb{N}$ there is an infinite set $X \subseteq \mathbb{N}$ such that $\mathrm{FS}(X)$ is monochromatic.

Let HT_k be the restriction of HT to k-colorings.

Proofs of HT

Hindman's original proof of HT was a complicated combinatorial argument.

Baumgartner (1974) and Towsner (2012) found simpler combinatorial arguments.

Furstenberg and Weiss (1978) proved HT using topological dynamics.

A very short "soft" proof was discovered by Galvin and Glazer (1977) using ultrafilters, specifically an *idempotent* ultrafilter under a certain binary operation which makes the set of ultrafilters on $\mathbb{N}$ a semigroup.

The book *Ramsey Theory* by Graham, Rothschild, and Spencer contains expositions of the Baumgartner, Furstenberg-Weiss, and Galvin-Glazer proofs.

Reverse Mathematics

Reverse Mathematics studies implications between sentences of second-order arithmetic over the *base theory* RCA_0.

The theory RCA_0 corresponds roughly to computable mathematics.

The theory WKL_0 is RCA_0 together with the statement that every infinite tree of finite binary strings has a path.

The theory ACA_0 is RCA_0 together with the arithmetic comprehension scheme.

The theory ACA_0^+ is RCA_0 together with the statement $(\forall X)[X^{(\omega)}$ exists]. One could think of $X^{(\omega)}$ as the collection of all arithmetical statements true of X.

In terms of logical strength, we have $\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ACA}_0^+$.

An upper bound on the strength of HT

A k-coloring c of $\mathbb{N}$ is called an *instance* of HT and an infinite set X such that $\mathrm{FS}(X)$ is monochromatic for c is called a *solution* for c. So HT is the statement that each instance c of HT has a solution X.

Theorem (Blass, Hirst, and Simpson, 1987)

If c is a computable instance of HT, then c has a solution X with $X \leq_T 0^{(\omega+1)}$. (Here $\leq_T$ denotes Turing reduciblity, and $0^{(\omega+1)}$ is the Turing jump of $0^{(\omega)}$.)

The proof of this result was based on Hindman's original proof of HT.

Corollary (Blass, Hirst, and Simpson)

Hindman's Theorem is provable in ACA_0^+.

The shorter proofs of HT do not appear to yield this corollary.

Lower Bounds on HT

Theorem (Blass, Hirst, and Simpson)

There is a computable instance c of HT such that all solutions X compute $0'$.

Their proof was an extremely ingenious parity argument.

Corollary (Blass, Hirst, and Simpson)

HT implies ACA_0 *in* RCA_0.

Theorem (Blass, Hirst, and Simpson)

There is a computable instance of HT with no Σ^0_2 *solution.*

Open Questions about Hindman's Theorem

(Q1) Is there a fixed $n \in \omega$ such that every computable instance of HT has a solution $X \leq_T 0^{(n)}$?

(Q1′) Is HT provable in ACA_0?

Note: A positive answer to (Q1′) implies a positive answer to (Q1).

A positive answer to (Q1) would likely lead to a positive answer to (Q1′’) as a corollary to the proof.

(Q2) Is there a computable instance c of HT such that every solution computes $0^{(\omega)}$?

(Q2′) Is HT equivalent to ACA_0^+ over RCA_0?

Bounded Versions of Hindman's Theorem

Hindman, Leader, and Strauss have asked:

Does every proof of Hindman's Theorem for sums of length at most 2 actually prove full Hindman's Theorem?

Reverse Mathematics suggests a formal analog of this question. For $X \subseteq \omega$, $n \in \mathbb{N}$, define

$$\mathsf{FS}^{\leq n}(X) = \{\Sigma(F) : F \subseteq X \ \&\ 0 < |F| \leq n\}$$

Let $\mathsf{HT}^{\leq n}_k$ be the assertion that for every k-coloring c of $\mathbb{N}$ there is an infinite set X such that $\mathsf{FS}^{\leq n}(X)$ is monochromatic. Let $\mathsf{HT}^{\leq n}$ be the statement $(\forall k)\mathsf{HT}^{\leq n}_k$.

A possible formalization of the informal question above is:

Does $\mathsf{HT}^{\leq 2}$ imply HT in RCA_0?

Lower bounds on bounded versions of HT

Theorem (Dzhafarov, Jockusch, Solomon, and Westrick)

There is a computable instance c of $HT_3^{\leq 3}$ *such that every solution X for c computes* $0'$.

This extends the corresponding result of Blass, Hirst, and Simpson for full HT. Our proof uses their very ingenious parity trick and has further ideas to bound the length of the relevant sums. In our coloring, the color of a number depends (among other things) on its expansion in base 7 .

Corollary (DJSW)

$HT_3^{\leq 3}$ *implies* ACA_0 *in* RCA_0.

Carlucci, Kołodzieczyk, Lepore, and Zdanowski studied modified versions of Hindman's Theorem in which the solution sets are required to meet certain natural sparseness conditions. Let D_n be the finite set with canonical index n, so $n = \sum_{i \in D_n} 2^i$. A set $S \subseteq \mathbb{N}$ satisfies the *apartness condition* if when $i, j \in S$ and $i < j$ we have $\max(D_i) < \min(D_j)$.

Carlucci et al showed that $\mathsf{HT}_2^{\leq 2}$ with the added condition that the solution must satisfy the apartness condition implies ACA_0 in RCA_0. From this they deduce:

Theorem

(Carlucci, Kołodzieczyk, Lepore, and Zdanowski) $HT_4^{\leq 2}$ implies ACA_0 in RCA_0.

Sums of length at most 2 with 2 colors

We know that $\mathsf{HT}_4^{\leq 2}$ and $\mathsf{HT}_3^{\leq 3}$ each imply ACA_0 over RCA_0. What if we use sums of length at most 2 and only 2 colors?

Theorem (DJSW)

Let A be a Δ^0_2 set. Then there is a computable instance c of $HT_2^{\leq 2}$ such that every solution X for c computes an infinite subset of A or $\overline{A}$.

Corollary

There is a computable instance of $HT_2^{\leq 2}$ with no computable solution, and hence $HT_2^{\leq 2}$ is not provable in RCA_0.

Ramsey's Theorem and Stability

Let $[A]^n$ be the set of n-element subsets of A.

Let RT^n_k be Ramsey's Theorem for k-colorings of n-element subsets of $\mathbb{N}$.

Thus RT^n_k asserts that for every coloring $c : [\mathbb{N}]^n \to k$ there is an infinite set $A \subseteq \mathbb{N}$ such that c is constant on $[A]^n$.

A coloring $c : [\mathbb{N}]^2 \to k$ is *stable* if for every a, $\lim_b c(a, b)$ exists.

Let SRT^2_2 be the restriction of RT^2_2 to stable colorings $c : [\mathbb{N}]^2 \to 2$.

In 2014, Chong, Slaman, and Yang proved that SRT^2_2 is strictly weaker than RT^2_2 over RCA_0. It remains open whether every ω-model of RCA_0+ SRT^2_2 is a model of RT^2_2.

Let $\mathsf{B}\Sigma^0_2$ be the Σ^0_2 bounding principle.

COROLLARY (DJSW)

In $\mathsf{RCA}_0 + B\Sigma^0_2$, $HT^{\leq 2}_2$ *implies* SRT^2_2.

The following questions are open. All implications considered are in RCA_0.

QUESTION

Does $\mathsf{HT}^{\leq 2}_2$ imply ACA_0?

QUESTION

Does ACA_0 imply $\mathsf{HT}^{\leq 2}_2$?

Sums of length exactly n

For $X \subseteq \mathbb{N}$ and $n \geq 1$ define:

$$\mathrm{FS}^{=n}(X) = \{\Sigma(F) : F \subseteq X \;\&\; |F| = n\}$$

So $\mathrm{FS}^{=n}(X)$ is the set of numbers of the form $x_1 + x_2 + \cdots + x_n$ with $x_1, x_2, \ldots, x_n$ distinct elements of X.
Let $\mathrm{HT}^{=n}_k$ be the assertion that for every k-coloring c of $\mathbb{N}$ there is an infinite set X such that $\mathrm{FS}^{=n}(X)$ is monochromatic.

$\mathrm{HT}^{=n}_k$ is simply the restriction of RT^n_k to instances c where $c(F)$ depends only on $\Sigma(F)$ for $F \in [\mathbb{N}]^n$. Thus

$$\mathsf{RCA}_0 \vdash (\forall n)(\forall k)[\mathrm{RT}^n_k \longrightarrow \mathrm{HT}^{=n}_k]$$

It follows that $\mathrm{HT}^{=n}_k$ is provable in ACA_0 for each **fixed** n and k.

Also, every computable instance of $\mathrm{HT}^{=n}_k$ has a Π^0_n solution, by the corresponding result for Ramsey's Theorem.

$\mathrm{HT}_2^{=2}$

We will focus on the most basic version of HT, namely $\mathrm{HT}_2^{=2}$. Since RT_2^2 does not imply ACA_0 by a result of Seetapun, it follows that $\mathrm{HT}_2^{=2}$ is strictly weaker than ACA_0 over RCA_0.

QUESTION:
Is $\mathrm{HT}_2^{=2}$ computably true? That is, does every computable instance have a computable solution?

This question remained open for several years. A positive result seemed hopeless, and attempted proofs of a negative result seemed to lead to a combinatorial swamp.

$\mathsf{HT}_2^{=2}$ IS NOT COMPUTABLY TRUE

THEOREM (CSIMA, DZHAFAROV, HIRSCHFELDT, J., SOLOMON, AND WESTRICK)

There is a computable instance of $HT_2^{=2}$ with no computable solution. Hence $HT_2^{=2}$ is not provable in RCA_0.

Define a computable coloring $c : \mathbb{N} \to 2$ meeting each requirement R_i, where R_i asserts that the ith c.e. set W_i is not a solution for c.

Strategy for R_i : Choose a suitably large number k_i, and, if $|W_i| \geq k_i$, let E_i consist of the first k_i numbers enumerated in W_i.

Ensure that for all sufficiently large s, $E_i + s$ is not monochromatic, where $E_i + s = \{k + s : k \in E_i\}$. (Note that $E_i + s \subseteq \mathsf{FS}^{=2}(E_i \cup \{s\})$ if $s \notin E_i$.)

This can be easily done for each fixed i, with $k_i = 2$. But even for two requirements significant conflicts arise.

Probabilistic methods

Try a probabilistic approach. Think of the values of $c(n)$ as mutually independent random variables. The probability that $E_i + s$ is monochromatic is small, namely 2^{-k_i+1}.

A naive argument breaks down because $\sum_s 2^{-k_i+1} = \infty$.

Instead, note that if $E_i + s$ and $E_j + t$ are disjoint, the events that these sets are monochromatic are independent. Furthermore, this happens “frequently".

We need a result that if $A_0, A_1, \ldots$ are “effectively given" events of “small probability" which are “frequently independent", there is a computable outcome c in which no A_i occurs.

Enter the Lovász Local Lemma and its effective version!

The Lovász Local Lemma

First consider the finite form. Let $[n] = \{1, 2, \ldots, n\}$.

Let $A_1, A_2, \ldots, A_n$ be events in a probability space. Suppose that for each $j \in [n]$ the set $N_j \subseteq [n]$ is such that A_j is independent of any intersection of events A_i for $i \in [n] \setminus N_j$. Suppose there are reals $r_1, r_2, \ldots, r_n$ in the interval $(0, 1)$ such that for all $j \leq n$

$$Pr(A_j) \leq r_j \, \Pi_{i \in N_j, i \neq j}(1 - r_i)$$

Then

$$Pr(\overline{A_1}\,\overline{A_2}\ldots\overline{A_n}) > 0$$

The infinite form of the LLL

This is similar to the finite form but applies to an infinite sequence of events, and concludes only that the intersection of their complements is nonempty. We assume that we have a topology on the space of outcomes.

Infinite form of LLL

Let $A_1, A_2, \dots$ be compact events in a probability space. Suppose that for each j the finite set $N_j \subseteq \mathbb{N}$ is such that A_j is independent of any intersection of events A_i for $i \in \mathbb{N} \setminus N_j$. Suppose there are reals $r_1, r_2, \dots$ in the interval $(0, 1)$ such that for all j

$$Pr(A_j) \leq r_j \, \Pi_{i \in N_j, i \neq j}(1 - r_i)$$

Then

$$\overline{A_1}\,\overline{A_2} \dots \neq \emptyset$$

The infinite form follows immediately from the finite form by compactness.

The setting for the Rumyantsev-Shen Theorem

Suppose that $x_1, x_2, \ldots$ are mutually independent $\{0,1\}$–valued random variables. Further, the probability that $x_n = k$ is a rational number effectively computable from n and k.

$A_1, A_2, \ldots$ are events such that each A_j is a finite Boolean combination of statements of the form $x_n = k$. Further, this Boolean combination is effectively computable from j.

For each j, $N(j)$ is finite, where $N(j)$ is the set of i such that the events A_i and A_j share at least one variable x_n in their definition. Further, a canonical index of $N(j)$ is effectively computable from j.

Note that each event A_j is independent of any Boolean combination of events A_i for $i \notin N(j)$.

The Rumyantsev-Shen Theorem

This is an effective version of the infinite version of the Lovász Local Lemma. Note that the hypothesis of the effective LLL is strengthened because the upper bound on $Pr(A_j)$ is multiplied by a constant $q < 1$.

Theorem (A. Rumyantsev and A. Shen)

Suppose there is a real number $q \in (0, 1)$ and a computable sequence of rational numbers $q_1, q_2, \ldots$ in $(0, 1)$ such that for all j,

$$Pr[A_j] \leq q\, q_j\, \Pi_{i \in N_j, i \neq j}(1 - q_i)$$

Then there is a **computable** *function $c : \mathbb{N} \to 2$ such that no event A_j occurs when $x_n = c(n)$ for all n.*

The proof is partly based on an efficient probabilistic algorithm due to Moser and Tardos for finding solutions to the finite version of the LLL. The use of this algorithm to prove an effective version of the LLL was suggested by Lance Fortnow.

A Corollary of the Rumyantsev-Shen Theorem

Rumyantsev and Shen proved a corollary to their result which is easily applicable and quickly yields the following result:

Corollary

For each rational number $r \in (0,1)$ *there is an integer* b *such that the following holds. Let* $F_0, F_1, \dots$ *be a computable sequence of finite sets, each of size at least* b*. Suppose that for each* $m \geq b$ *and* n*, there are at most* 2^{rm} *many* j *such that* $|F_j| = m$ *and* $n \in F_j$*, and we can computably determine a canonical index for the set of all such* j *given* m *and* n*. Then there is a computable* $c : \mathbb{N} \to 2$ *such that for each* j *the set* F_j *is not monochromatic for* c*.*

The proof that $HT_2^{=2}$ is not computably satisfiable

Let b be as in the last corollary for $r = 1/2$ and assume also that $m \leq 2^{m/2}$ for all $m \geq b$. Let $k_i = i + b$. If $|W_i| \geq k_i$, let E_i consist of the first k_i numbers enumerated in W_i. It suffices to show the existence of a computable coloring $c : \mathbb{N} \to 2$ such that, for all i, if $|W_i| \geq k_i$, then for all sufficiently large s, $E_i + s$ is not monochromatic.

Let $F_0, F_1, \ldots$ be a computable enumeration without repetition of all sets of the form $E_i + s$ where W_i has at least k_i elements at stage s (so E_i is known at stage s).

Apply the previous corollary to the sequence $F_0, F_1, \ldots$ to obtain a computable coloring $c : \mathbb{N} \to 2$ with no F_j monochromatic. To check the hypotheses, note that all sets in the sequence $F_0, F_1, \ldots$ of a given size m are translates of the **fixed** set E_i, where $k_i = m$, and n belongs to at most $|E_i| = m$ translates of E_i.

Then if W_i is infinite, then some F_j has the form $E_i + s$ with $s \in W_i \setminus E_i$, so W_i is not a solution for c.

An Extension

Theorem

There is a computable instance c of $HT_2^{=2}$ with no Σ_2^0 solution.

The proof is similar to the Σ_1^0 case, except that W_i is replaced everywhere by W_i^K. Also, a K-oracle is needed to compute E_i when it exists. By the Limit Lemma, E_i can be computably approximated. Let $E_{i,s}$ be the approximation at stage s to E_i. Thus, if E_i exists, we have $E_{i,s} = E_i$ for all sufficiently large s. Without loss of generality, assume that $|E_{i,s}| = k_i$ for all i and s.

Make the sequence $F_0, F_1, \ldots$ much as before, but ensure that if $E_{i,s} + s$ is put into the sequence, then it is disjoint from all sets in the sequence of the form $E_{i,t} + t$ for $t < s$ with $E_{i,t} \neq E_{i,s}$. Then, for any numbers n and m, the all sets in the sequence of size m which contain n are translates of a **fixed** set, and the counting argument works as before.

$HT_2^{=2}$ AND WKL_0

COROLLARY

Neither of $HT_2^{=2}$ and WKL_0 implies the other over RCA_0.

$HT_2^{=2}$ does not imply WKL_0 by Liu's remarkable theorem that RT_2^2 does not imply WKL_0.

WKL_0 does not imply $HT_2^{=2}$ because WKL_0 has an ω-model containing only low sets, but RCA_0+ $HT_2^{=2}$ does not, by the Σ_2^0 theorem.

$\mathsf{HT}_2^{=2}$ and the Rainbow Ramsey Theorem

Let RRT_2^2 be the assertion that if $c : [\mathbb{N}]^2 \to \mathbb{N}$ and $|c^{(-1)}(n)| \leq 2$ for all $n \in \mathbb{N}$, there is an infinite set X such that the restriction of c to $[X]^2$ is injective. This is a special case of the Rainbow Ramsey Theorem. Csima and Mileti showed that RRT_2^2 is strictly weaker than RT_2^2 over RCA_0.

Theorem (CDHJSW)

$HT_2^{=2}$ implies RRT_2^2 over RCA_0.

A function f is DNC relative to a set X if $(\forall e)[f(e) \neq \Phi_e^X(e)]$. The principle 2-DNC asserts that for every X there is a function which is DNC relative to X'. It follows easily from the proof of the Σ_2^0 theorem that there is a computable instance of $\mathsf{HT}_2^{=2}$ such that all solutions compute functions which are DNC relative to $0'$. A careful analysis of this proof shows that $\mathsf{HT}_2^{=2}$ implies 2-DNC. Joe Miller has shown that 2-DNC is equivalent to RRT_2^2, so the theorem follows.

Some Open Questions

All implications considered are over RCA_0.

Question

Does $\mathsf{HT}^{=2}_2$ imply RT^2_2 ?

Question

Does RRT^2_2 imply $\mathsf{HT}^{=2}_2$?

Question

For $n \geq 3$, is there a computable instance of $\mathsf{HT}^{=n}_2$ with no Σ^0_n solution?

Question

For which n and k does $\mathsf{HT}^{=n}_k$ imply ACA_0?