Generalizing VC dimension to higher arity

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Suppose $G = (V, E)$ is a finite graph, so V is a finite set and $E \subseteq [V]^2$ is a set of pairs.

It is natural to put the *counting measure* on subsets of V^k:

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Various results in extremal combinatorics (Szemerédi regularity, graph removal, etc) can be viewed as probabilistic theorems in this setting. For example:

Theorem (Triangle Removal)

For every $\epsilon > 0$ there is a $\delta > 0$ such that either:

- *the set of triangles has measure $> \delta$, or*
- *there is a set $R \subseteq E$ of edges with measure $< \epsilon$ such that $(V, E \setminus R)$ has no triangles.*

Suppose that, for each i, $G_i = (V_i, E_i)$ is a graph with $|V_i|$ finite and $\lim_{i\to\infty} |V_i| = \infty$.

The ultraproduct

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We can hope to lift the counting measure on the V_i^k to a measure on V:

- when $X_i \subseteq V_i^k$ for all i, there is an internal set $X = \prod_{\mathcal{U}} X_i$, and we can define

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- we can let $\mathcal{B}_k$ be the σ-algebra generated by the internal subsets of V^k.

To summarize:

- we have a graph (V, E) with uncountably many vertices,
- for each k, we have a measure space $(V^k, \mathcal{B}_k, \mu^k)$,
- for internal sets (like E, or the set of triangles), the measure in μ^k is the limit of the corresponding measures μ_i^k.

To summarize:

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For example, one way to prove triangle removal is to prove:

Theorem

If (V, E) is an ultraproduct of graphs, either:

- *the set of triangles has positive measure, or*
- *for every $\epsilon > 0$, there is an internal set $R \subseteq E$ with measure $< \epsilon$ such that $(V, E \setminus R)$ has no triangles.*

But the measurable sets of k-tuples are *not* the product of the sets of singletons:

Theorem

There is a set $A \in \mathcal{B}_2$ which is not in the σ-algebra generated by $\mathcal{B}_1 \times \mathcal{B}_1$.

Recall that $\mathcal{B} \times \mathcal{B}$ is the σ-algebra generated (under complements and countable unions and intersections) by rectangles $B \times C$.

That means sets in $\mathcal{B}_1 \times \mathcal{B}_1$ are *approximated by rectangles*:

Theorem

If $A \in \mathcal{B}_1 \times \mathcal{B}_1$ *then, for every* $\epsilon > 0$, *there exist* $B_i, C_i \in \mathcal{B}_1$ *so that*

$$\mu(A \triangle (\bigcup_{i \leq k} B_i \times C_i)) < \epsilon.$$

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On the other hand, the sets in $\mathcal{B}_2 \setminus (\mathcal{B}_1 \times \mathcal{B}_1)$ *cannot* be approximated in this way.

In fact, any set has a decomposition

$$\chi_A \approx f(x, y) + \sum_{i \leq d} \gamma_i \chi_{B_i}(x) \chi_{C_i}(y)$$

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This phenomenon is familiar in finite combinatorics: the product $\mathcal{B}_1 \times \mathcal{B}_1$ corresponds to the partition given by Szemerédi regularity.

Definition

A graph $E \subseteq V^2$ has *VC dimension* $\geq d$ if there exist elements

$$y_1, \ldots, y_d \in V$$

such that, for every $S \subseteq \{y_1, \ldots, y_d\}$, there is some $x \in V$ so that

$$E_x \cap \{y_1, \ldots, y_d\} = S.$$

So the slices E_x are able to pick out every subset of the set $\{y_1, \ldots, y_d\}$.

Example

Consider the graph $E \subseteq [0,1] \times [0,1]$ where $(i,j) \in E$ iff $i < j$.

This has VC dimension 2: given any $y_1, y_2 \in [0,1]$, without loss of generality $y_1 < y_2$. Then, no matter what x is,

$$E_x \cap \{y_1, y_2\} \text{ is one of } \emptyset, \{y_2\}, \{y_1, y_2\}.$$

Example

If E_i is a random graph on V_i and $(V, E) = \prod_{\mathcal{U}}(V_i, E_i)$ then the VC dimension of E is infinite.

Given any $\{y_1, \ldots, y_d\} \subseteq V$ and any $S \subseteq \{y_1, \ldots, y_d\}$, the probability that $E_x \cap \{y_1, \ldots, y_d\} \neq S$ is $1 - 2^{-d}$, so if we have n choices of x, by the union bound, the probability the VC dimension is $\leq d$ is bounded by

$$2^d(1 - 2^{-d})^n$$

which approaches 0 as n approaches ∞.

Standard facts about VC dimension:

Theorem

- *VC dimension is symmetric up to some loss of constants: if* $E \subseteq X \times Y$ *has VC dimension* $\leq d$ *then the flipped graph* $E' \subseteq Y \times X$ *has VC dimension* $\leq 2^{d+1} - 1$.

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- *Sauer-Shelah: If* E *has VC dimension* $< d$ *then whenever* $\{y_1, \ldots, y_m\} \in Y$, *there are at most* $\sum_{i=0}^{d-1} \binom{m}{i}$ *sets* $S \subseteq \{y_1, \ldots, y_m\}$ *such that there is an* x *with*

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The quantity in Sauer-Shelah is a polynomial, so Sauer-Shelah says:

> *The number of subsets of* $\{y_1, \dots, y_m\}$ *is either bounded by a polynomial or (for some sets* $\{y_1, \dots, y_m\}$*) contains every subset (and therefore grows exponentially).*

Theorem (The VC Theorem)

Suppose $E \subseteq X \times Y$ has finite VC dimension and let $\epsilon > 0$. Then there exists a set $\{y_1, \dots, y_m\}$ (with m depending only on the VC dimension and ϵ) such that for every single $x \in X$, either:

- $\mu(E_x) < \epsilon$, *or*
- $|E_x \cap \{y_1, \dots, y_m\}| \neq \emptyset$.

The set $\{y_1, \dots, y_m\}$ is called an ϵ-net.

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The set $\{y_1, \ldots, y_m\}$ is called an ϵ-net.

- Almost every set $\{y_1, \ldots, y_m\}$ has this property.
- By choosing $\{y_1, \ldots, y_m\}$ slightly larger, we can ensure

$$\frac{|E_x \cap \{y_1, \ldots, y_m\}|}{m} \approx \mu(E_x).$$

This is called an ϵ-approximation.

Corollary

Suppose $E \subseteq X \times Y$ *has finite VC and let* $\epsilon > 0$. *Then there exists* $\{y_1, \ldots, y_m\} \subseteq Y$ *so that for any* $x, x' \in X$, *either:*

- $\mu(E_x \triangle E_{x'}) < \epsilon$, *or*
- $E_x \cap \{y_1, \ldots, y_m\} \neq E_{x'} \cap \{y_1, \ldots, y_m\}$.

Corollary

Suppose $E \subseteq X \times Y$ has finite VC and let $\epsilon > 0$. Then there exists $\{y_1, \dots, y_m\} \subseteq Y$ so that for any $x, x' \in X$, either:

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Corollary

Suppose $E \subseteq X \times Y$ has finite VC dimension and let $\epsilon > 0$. Then there exist $x_1, \dots, x_k$ such that, for every $x \in X$, there is some x_i with $\mu(E_x \triangle E_{x_i}) < \epsilon$.

Corollary (Regularity for VC dimension)

If $E \subseteq V^2$ has finite VC dimension then:

- *E belongs to $\mathcal{B}_1 \times \mathcal{B}_1$,*
- *the number of rectangles needed to approximated E to within ϵ is bounded by a polynomial in $1/\epsilon$.*

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Sketch.

Choose $x_1, \ldots, x_k$ so that, for every $x \in X$, there is some x_i with $\mu(E_x \triangle E_{x_i}) < \epsilon|X|$. Take $X_i = \{x \mid |E_x \triangle E_{x_i}| < \epsilon\}$.

For each $S \subseteq \{x_1, \ldots, x_k\}$, take $Y_S = \{y \mid (x_i, y) \in E \text{ iff } x_i \in S\}$.

Then

$$f = \sum_{i,S} \frac{\mu(E \cap (X_i \times Y_S))}{\mu(X_i \times Y_S)} \chi_{X_i} \chi_{Y_S}$$

suffices. □

The same ideas apply to 3-graphs (that is, hypergraphs whose edges are triples):

- when each (V_i, H_i) is a 3-graph with $|V_i|$ finite and $\lim_{i\to\infty}|V_i| = \infty$, the ultraproduct (V, H) is a k-graph on an uncountable set,

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- when $X_i \subseteq V_i^k$ for all i, there is a set $X = \prod_{\mathcal{U}} X_i$ with $\mu^k(X) = \lim_{\mathcal{U}} \frac{|X_i|}{|V_i|^k}$,

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- when $X_i \subseteq V_i^k$ for all i, there is a set $X = \prod_{\mathcal{U}} X_i$ with $\mu^k(X) = \lim_{\mathcal{U}} \frac{|X_i|}{|V_i|^k}$,
- for each k, we have a measure space $(V^k, \mathcal{B}_k, \mu^k)$.

Example

Choose $E, F, G \in \mathcal{B}_2$ to be quasi-random.

We define

$$H = \{(x, y, z) \mid \chi_E(x, y) + \chi_F(x, z) + \chi_G(y, z) \in \{1, 3\}\}.$$

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This hypergraph is “random” relative to any box:

$$\mu(H \cap (A \times B \times C)) \approx \frac{1}{2}\mu(A \times B \times C).$$

Certainly $\mathcal{B}_3$ contains sets not in $\mathcal{B}_1 \times \mathcal{B}_1 \times \mathcal{B}_1$, or even in $\mathcal{B}_2 \times \mathcal{B}_1$.

But these do not exhaust the ways lower-order sets could define sets of triples. We need to consider *cylinder intersections*: sets of the form

$$\{(x, y, z) \mid (x, y) \in A \text{ and } (x, z) \in B \text{ and } (y, z) \in C\}$$

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where $A, B, C \in \mathcal{B}_2$.

These sets generate a σ-algebra $\mathcal{B}_{3,2}$. We still have $\mathcal{B}_3 \supsetneq \mathcal{B}_{3,2}$.

The appropriate decomposition is to take a hypergraph χ_E and write it in the form

$$\chi_E(x,y,z) \approx f(x,y) + \sum_{i \leq d_2} \gamma_i \chi_{A_i}(x,y)\chi_{B_i}(x,z)\chi_{C_i}(y,z) + \sum_{i \leq d_1} \delta_i \chi_{D_i i}(x)\chi_{F_i}(y)\chi_{G_i}(z)$$

where A_i, B_i, C_i are quasi-random (possibly directed) graphs.

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This does correspond in a precise way to hypergraph regularity for 3-graphs, but the correspondence is a bit more complicated because the interactions of different bounds are more complicated.

Definition

A 3-graph $H \subseteq V^3$ has 2-VC dimension $\geq d$ if there is a rectangle

$$y_1, \ldots, y_d \in V, \; z_1, \ldots, z_d \in V$$

such that, for every $S \subseteq \{y_1, \ldots, y_d\} \times \{z_1, \ldots, z_d\}$, there is some $x \in V$ so that

$$H_x \cap (\{y_1, \ldots, y_d\} \times \{z_1, \ldots, z_d\}) = S.$$

Example

Recall the hypergraph where $E, F, G \subseteq V^2$ are each quasi-random, and H consists of those (x, y, z) so that an odd number of the pairs $(x, y), (x, z), (y, z)$ belong to the respective graphs.

We claim H has 2-VC dimension ≤ 65.

Example

Recall the hypergraph where $E, F, G \subseteq V^2$ are each quasi-random, and H consists of those (x, y, z) so that an odd number of the pairs $(x, y), (x, z), (y, z)$ belong to the respective graphs.

We claim H has 2-VC dimension ≤ 65. Consider any $\{y_1, \ldots, y_5\} \subseteq V$ and $\{z_1, \ldots, z_{65}\} \subseteq V$. By Ramsey's Theorem (and possibly reordering the elements), without loss of generality we may assume either $\{y_1, y_2, y_3\} \times \{z_1, z_2, z_3\} \subseteq G$ or $\{y_1, y_2, y_3\} \times \{z_1, z_2, z_3\} \cap G = \emptyset$.

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Then no x can have

$$H_x \cap (\{y_1, y_2, y_3\} \times \{z_1, z_2, z_3\}) = \{(y_1, z_1), (y_2, z_2), (y_3, z_3)\} :$$

this would imply that no two of $\chi_E(x, y_1), \chi_E(x, y_2)$, and $\chi_E(x, y_3)$ can be equal, which is impossible.

Theorem (Chernikov-Palacin-Takeuchi)

If H has VC dimension $< d$ then whenever $\{y_1, \ldots, y_m\} \subseteq V$ and $\{z_1, \ldots, z_m\} \subseteq V$, there is an $\epsilon(d) > 0$ so that there are at most $2^{m^{2-\epsilon(d)}}$ sets $S \subseteq \{y_1, \ldots, y_d\} \times \{z_1, \ldots, z_d\}$ such that there is an x with

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The bound $2^{m^{2-\epsilon}}$ is not as strong as original conjectured, but CPT show it is close to optimal.

The key theorem about VC dimension was:

Theorem (The VC Theorem)

Suppose $E \subseteq V^2$ has finite VC dimension and let $\epsilon > 0$. Then there exists a set $\{y_1, \ldots, y_m\}$ (with m depending only on the VC dimension and ϵ) such that for every single $x \in X$, either:

- $\mu(E_x) < \epsilon$, *or*
- $|E_x \cap \{y_1, \ldots, y_m\}| \neq \emptyset$.

We don't even know what the right definition of an ϵ-net would be for 2-VC dimension.

Theorem

Suppose $E \subseteq V^2$ has finite VC dimension and let $\epsilon > 0$. Then there exist $x_1, \ldots, x_n$ such that, for every $x \in V$, there is some x_i with $\mu(E_x \triangle E_{x_i}) < \epsilon$.

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Theorem (Chernikov-T.)

Suppose $H \subseteq V^3$ has finite 2-VC dimension and let $\epsilon > 0$. Then there exist $x_1, \ldots, x_n$ such that, for every $x \in V$, there is a partition

$$V^2 = \bigcup_{j \leq m, k \leq m} B_j \times C_k$$

and, for each pair (j, k), a Boolean combination $E^{(j,k)}$ of the E_{x_i}, such that

$$|\mu(E_x \triangle \bigcup_{j \leq m, k \leq m} E^{(j,k)} \cap (B_j \times C_k)) < \epsilon.$$

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The end.