Revisiting the canonical Erdős-Rado theorem

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Ramsey theory in logic, combinatorics and complexity

Outline

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- The finite canonical Erdős-Rado theorem.
- Canonical colorings on Fraïssé structures.
- Results.

Part I

The finite canonical Erdős-Rado theorem

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Theorem (Erdős-Rado, 50)

Let $m \leq n \in \mathbb{N}$, $\chi : \binom{\mathbb{N}}{m} \to \mathbb{N}$.
Then there is $\tilde{B} \in \binom{\mathbb{N}}{n}$ such that χ is canonical on $\binom{\tilde{B}}{m}$ i.e.

$$\exists I \subset m \quad \forall a, a' \in \binom{\tilde{B}}{m} \quad \chi(a) = \chi(a') \Leftrightarrow \mathrm{proj}_I(a) = \mathrm{proj}_I(a')$$

In words: Any coloring is essentially a projection when suitably localized.

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Remark

When $I = \emptyset$, χ *is constant.*
Conversely, $I = \emptyset$ *is the only possible canonization when* χ *has finite range.*

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- *Holds for various classes of graphs [Dobrinen-Mijares-Trujillo, 17]... rediscovering a result of Prömel-Voigt from 85!*
- *Recently proved for finite ordered tournaments and finite posets ordered with linear extensions (Mašulović, preprint 17).*

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- *How frequent are such results in structural Ramsey/Fraïssé theory?*
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- Any finite Ramsey theorem in the Fraïssé context admits a canonical Erdős-Rado counterpart...
- ... But finding out what this counterpart is is not Ramsey theory anymore.

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Goal of today's talk:

- Any finite Ramsey theorem in the Fraïssé context admits a canonical Erdős-Rado counterpart...
- ... But finding out what this counterpart is is not Ramsey theory anymore.
- In addition, it seems that there is not more to it than extreme amenability.

Part II

Canonical colorings

Definition

Let $m \in \mathbb{N}$. A coloring $\chi : \binom{\mathbb{N}}{m} \to \mathbb{N}$ is canonical when the equivalent relation it induces on $\binom{\mathbb{N}}{m}$ is S_∞-invariant, where

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Theorem (Erdős-Rado, 50 ; V2)

Let $m \leq n \in \mathbb{N}$. *Then:*

1. $\forall \chi : \binom{\mathbb{N}}{m} \to \mathbb{N} \quad \exists \tilde{B} \in \binom{\mathbb{N}}{m} \quad \exists c\, canonical \quad \chi \upharpoonright \binom{\tilde{B}}{m} = c \upharpoonright \binom{\tilde{B}}{m}$
2. *Up to a renaming of its range, any canonical coloring is a projection.*

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2. *Up to a renaming of its range, any canonical coloring is a projection.*

It is under that form that the canonical Erdős-Rado theorem will generalize to the Fraïssé context. Possibly, the class of canonical colorings will be larger than just the set of projections.

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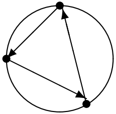
Examples

$\mathbb{N}$, $(\mathbb{Q}, <)$, *the random graph, the generic countable* K_n*-free graph, the countably-dimensional vector space over a given finite field, the countable atomless Boolean algebra, the generic countable poset, the dense local order* $S(2)$*:*

- *Vertices: Rational points of* S^1 *in complex plane (no opposite points).*
- *Arcs:* $x \to y$ *iff (counterclockwise angle from* x *to* y*)* $< \pi$.



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- *$\mathbb{F}$ has the Ramsey property when:*
  for any finite $A, B \subset \mathbb{F}$, any finite coloring of $\binom{\mathbb{F}}{A}$, there is $\tilde{B} \cong B$ where all embeddings of A have same color.

Examples

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- *Now many more by: Aranda et al., Bartosova-Kwiatkowska, Bartosova-Lopez-Abad-Mbombo, Bodirsky, Dorais et al., Foniok, Foniok-Böttcher, Jasiński, Jasiński-Laflamme-NVT-Woodrow, Kechris-Sokić, Kechris-Sokić-Todorcevic, Kwiatkowska, Nešetřil, Nešetřil-Hubička, NVT, Sokić, Solecki, Solecki-Zhao,...*

Part III

Results

Theorem (NVT, 17)

Let $\mathbb{F}$ be Fraïssé with the Ramsey property, and $A, B \subset \mathbb{F}$ finite. Then:

$$\forall \chi : \binom{\mathbb{F}}{A} \to \mathbb{N} \quad \exists b \in \binom{\mathbb{F}}{B} \quad \exists c \, canonical \quad \chi \upharpoonright \binom{b(B)}{A} = c \upharpoonright \binom{b(B)}{A}$$

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- The natural step at that point is... to ask your favorite model theorist... who will tell you that there is no general method for such a task, and that it could be truly difficult.
- Still, there are some natural conditions under which
  - there are only finitely many such relations.
  - the projections are the only canonical colorings.

Definition

Let $A, B \subset \mathbb{F}$ *finite. A* ***joint embedding*** *of* A *and* B *is a pair of embeddings of* A *and* B *into some finite* $C \subset \mathbb{F}$ *such that* $C = a(A) \cup b(B)$.

Remark

There is a natural notion of isomorphism between two such things.

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Proposition

Let $\mathbb{F}$ be Fraïssé, $A \subset \mathbb{F}$ finite. Assume that there are only finitely many isomorphism types of joint embeddings of two copies of A. Then:
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Corollary

Assume that $\mathrm{Aut}(\mathbb{F})$ *is Roelcke precompact (e.g.* $\mathbb{F}$ *has finite language, or is* $\aleph_0$*-categorical).*
Then, for every finite $A \subset \mathbb{F}$*, and up to a renaming of the range, there are only finitely many canonical colorings of* $\binom{\mathbb{F}}{A}$*.*

Theorem (NVT, 17)

Let $\mathbb{F}$ be Fraïssé, satisfying the free amalgamation property. Then, up to a renaming of the range, the canonical colorings are exactly the projections.

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Question

When the canonical colorings are the projections, the group $\mathrm{Aut}(\mathbb{F})$ is topologically simple. What about the converse?

NB: When $\mathbb{F}$ has free amalgamation, $\mathrm{Aut}(\mathbb{F})$ is top. simple provided it is not Sym($\mathbb{F}$) and it acts transitively on $\mathbb{F}$ (McPherson-Tent, 11).