Graphs and Combinatorics (1993) 9: 63–73

Graphs and Combinatorics

P-Partitions and the Plactic Congruence*

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Abstract. A conjecture of Stanley states that if the generating function of a poset P is symmetric, then in fact P belongs to the family of posets induced by some skew shape λ/μ. In this paper we show that if the set $L(P)$ of the linear extensions of a poset P is plactic-closed, then P is a poset induced by a skew shape.

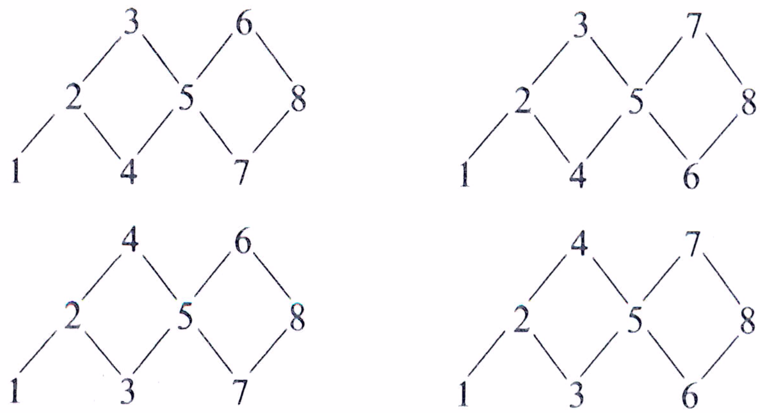
Let P be an alphabet consisting of n letters, with a total order $\leq$, usually thought of as $[n]$, and let $\leq_P$ be a partial order on P. As usual P^* indicates the free monoid on the alphabet P. Let $L(P)$ be the set of words of P^* which are a linear refinement (or linear extension) of P: that is, $w \in L(P)$ if

(i) w is a *standard* word of length n (i.e., in w all the letters of P occur without repetition)

(ii) whenever $x \leq_P y$ then x appears on the left of y in $w = \ldots x \ldots y \ldots$

Given a skew shape λ/μ of weight $|\lambda/\mu| = n$ and a totally ordered alphabet P, a family $P_{\lambda/\mu}$ of posets can be defined by filling the skew shape λ/μ with the letters of P in increasing order (with respect to $\leq$) on the rows, and in decreasing order on the columns, and then by rotating it 45° counterclockwise.

For example, if $P = [8]$ with the natural total order, and $\lambda/\mu = 443/21$ then $P_{\lambda/\mu}$ contains the following posets:



For a standard word w on P define its composition $\mathscr{C}(w)$ as the composition

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$$(|u_1|, \ldots, |u_k|)$$

where $w = u_1 \ldots u_k$, each u_i is an increasing word and k is minimal. For a composition $\mathscr{C} = (c_1, \ldots, c_k)$ of n, define the function of infinitely many variables:

$$F_c = \sum_{x_1 \le \cdots \le x_{c_1} < x_{c_1+1} \le \cdots \le x_{c_1+c_2} < x_{c_1+c_2+1} \le \cdots \le x_n} x_1 x_2 \ldots x_n$$

For example, if $w = 312$ then $u_1 = 3$, $u_2 = 12$, $\mathscr{C}(w) = (1, 2)$ and $F_{(1\,2)} = \sum_{x<y\le z} xyz$. To any subset L of permutations, we can associate the function

$$F_L = \sum_{w \in L} F_{\mathscr{C}(w)}$$

and to the poset P, the function

$$\Gamma(P) = F_{L(P)} = \sum_{w \in L(P)} F_{\mathscr{C}(w)}$$

Remark 1. This definition of $\Gamma(P)$ is equivalent to that of the generating function for the poset $(P, \le_P)$ given via P-partitions, and is defined as follows [1]: denote by $A(P)$ the set of all P-partitions, i.e., the functions $f \colon P \to \mathbf{N}$ such that, if $i \le_P j$ then $f(i) \le f(j)$, and if $i \le_P j$ but $i > j$ (in the total order of P) then $f(i) < f(j)$. Then

$$\Gamma(P) = \sum_{f \in A(P)} x_{f(1)} x_{f(2)} \ldots x_{f(m)}.$$

In [6] Stanley conjectures that *if $\Gamma(P)$ is a symmetric function, then there exist λ, μ such that $P \in \mathbf{P}_{\lambda/\mu}$.*

Recall [4] that the plactic congruence $\sim$ on the set of words in P^* is defined by Knuth's relations [3]:

$$\begin{array}{llll} \text{if} & x < y < z & \text{then} & yzx \sim yxz \\ & & \text{and} & zxy \sim xzy \\ \text{if} & x < y & \text{then} & yxx \sim xyx \\ & & \text{and} & yyx \sim yxy \end{array}$$

That is, we interchange the order of two letters whenever they are preceeded or followed by a letter that is in between.

The following facts motivate the consideration of the plactic congruence:

Fact 1. *If $P \in \mathbf{P}_{\lambda/\mu}$, then $L(P)$ is plactic-closed.*

This may be seen directly from the definitions, and is proved in [2].

Fact 2. *If L is a set of permutations which is plactic-closed, then F_L is a symmetric function.*

This result may be obtained by a standard application of the Robinson-Schensted correspondance.

Now, suppose that Stanley's Conjecture is true. If $L(P)$ is plactic closed, then $\Gamma(P)$ is a symmetric function by Fact 2, and the Conjecture states that $P \in \mathbf{P}_{\lambda/\mu}$. In this paper we give a direct proof of this result:

Theorem 1. *If $L(P)$ is plactic-closed, then $P \in \mathbf{P}_{\lambda/\mu}$ for some skew diagram λ/μ, i.e. it is one of the orders induced by a skew diagram λ/μ.*

To prove this theorem, we will study the Hasse diagram of the poset locally, giving some restrictions to its subposets.

It is easy to see that every linear refinement of a subset Q of P appears as a subword of some linear refinement of P, but in general not as a factor.

Recall that a subposet Q of a poset P is *convex* if, whenever $p \in P$ is such that $q_1 \leq_P p \leq_P q_2$, for some $q_1, q_2 \in Q$, then $p \in Q$.

In this particular case we can state the following:

Lemma 1. *If $Q \subseteq P$ is a convex subposet of a poset P then, for each $w \in L(Q)$, there exists $\bar{w} \in L(P)$ such that w is a factor of $\bar{w}$. That is, any linear extension of Q appears as a factor of some linear extension of P.*

Proof. By induction on $|P\backslash Q|$. Let A, B, C be the following subsets of $P\backslash Q$

$$A = \{x \in P\backslash Q \mid \exists q \in Q \text{ such that } x \leq_P q\}$$

$$B = \{x \in P\backslash Q \mid \exists q \in Q \text{ such that } x \geq_P q\}$$

$$C = \{x \in P\backslash Q \mid x \text{ is not comparable to any element of } Q\}$$

Clearly $A \cup B \cup C = P\backslash Q$ and $A \cap C = B \cap C = \emptyset$.

Furthermore we have also $A \cap B = \emptyset$: if not, let $y \in A \cap B$, then there exist $q \in A$, $q' \in B$ with $q' \leq_P y \leq_P q$, and the convexity of Q implies $y \in Q$, a contradiction. So A, B, C form a partition of $P\backslash Q$.

If $|P\backslash Q| = 0$, then $Q = P$ and the statement is trivially true.

Now suppose $|P\backslash Q| > 0$.

If $C \neq \emptyset$, let $c \in C$. Now $Q' = Q \cup \{c\}$ is still convex in P: indeed there is no element $p \in P$, $q \in Q$ such that $c \leq_P p \leq_P q$ or $q \leq_P p \leq_P c$, otherwise c would be comparable with Q. We have $cw \in L(Q')$; since $|P\backslash Q'| < |P\backslash Q|$, by induction, there exist u, v such that $\bar{w} = u(cw)v = (uc)wv \in L(P)$.

If $C = \emptyset$, and $A \neq \emptyset$, let $a \in A$ be maximal with respect to the partial order $\leq_P$. In this case $Q' = Q \cup \{a\}$ is convex in P: otherwise there exist $p \in P\backslash Q$, $q \in Q$ with $a \leq_P p \leq_P q$, so that p would be in A and $p \geq_P a$, contradicting the maximality of a. Furthermore we cannot have $q \leq_P p \leq_P a$, because by definition of A there exists $q' \in Q$ such that $a \leq_P q'$, so $q \leq_P a \leq_P q'$ and the convexity of Q says $a \in Q$. Now observe that $aw \in L(Q')$: if not, there exists $q \in Q$ with $q \leq_P a$, meaning that $a \in A \cap B$, impossible. Since $|P\backslash Q'| < |P\backslash Q|$, by induction there exist u, v such that $\bar{w} = u(aw)v \in L(P)$.

Finally suppose $C = \emptyset$, $A = \emptyset$, and $B \neq \emptyset$, let b be a minimal element in B. A similar argument as above shows that $Q' = Q \cup \{b\}$ is convex in P and $wb \in L(Q')$, so that wb is a factor of some word of $L(P)$, and the lemma is proved. □

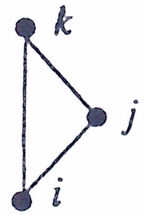
Corollary 1. *If Q is convex in P and $L(P)$ is plactic-closed, then $L(Q)$ is plactic-closed.*

Proof. Denote by π: $P^* \to Q^*$ the canonical projection of the set of words on the alphabet P onto the set of words on Q, which to any $u \in P^*$ associates the word

$\pi(u)$ of Q^* obtained by erasing in u the letters of $P \backslash Q$. It is clear, by the definition of a linear refinement, that if $u \in L(P)$ then $\pi(u) \in L(Q)$. Let $w \in L(Q)$ and $w \sim w'$: we want to show that $w' \in L(Q)$. By Lemma 1, we can find $\overline{w} = uwv \in L(P)$. We have $\overline{w} \sim uw'v$ (as $\sim$ is a congruence) and $uw'v \in L(P)$ since $L(P)$ is plactic-closed; in particular its projection $\pi(uw'v) = w'$ is in $L(Q)$. $\square$

Denote by $\lessdot_P$ the covering relation of the partial order $\leq_P$, that is $i \lessdot_P j$ if and only if $i \leq_P j$ and no element $k \in P$ satisfies $i \leq_P k \leq_P j$. Recall that the *Hasse diagram* of a poset P is the graph whose vertices are the elements of P and whose edges are the covering relations, such that if $i \leq_P j$ then j is drawn above i. For instance, if P is given by $i \leq_P j, j \leq_P k, i \leq_P k$, then its Hasse diagram is

and not

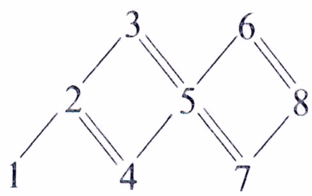


We adopt the following convention: if $i \lessdot_P j$ we draw

if $i < j$

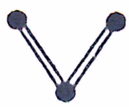
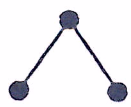
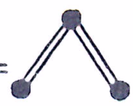
if $i > j$

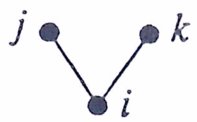
So the Hasse diagram of the first poset in the example will look like



From now on we suppose that $L(P)$ is plactic-closed.

Lemma 2. *None of the following configurations appear in the Hasse diagram of* P:

$Q_1 =$  $Q_2 =$  $Q_3 =$  $Q_4 =$ 

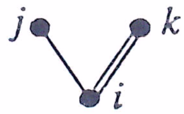
Proof. First remark that all of the Q_i's are convex subposets of P, because we are looking at the Hasse diagram. Now suppose that $Q_1 =$ 

is in the Hasse diagram of P.

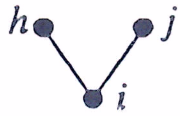
This means $i \leq_P j$ with $i < j$ and $i \leq_P k$ with $i < k$. If $j < k$ [resp. $k < j$] we have $i < j < k$ [resp. $i < k < j$] so $ikj \sim kij$ [resp. $ijk \sim jik$] with $ikj \in L(Q_1)$, but $kij \notin L(Q_1)$ [resp. $ijk \in L(Q_1)$, but $jik \notin L(Q_1)$], a contradiction, as $L(Q_1)$ is plactic closed by Corollary 1. Similarly, one can prove that Q_2, Q_3, and Q_4 do not appear in P. □

Lemma 3. *For any $i \in P$, there can be at most two elements covering i and at most two elements which are covered by i.*

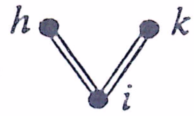
Proof. Suppose that $j, k >_P i$; then by Lemma 2 we must have



For any other $h \in P$ which covers i, we would have



or else



as subposet of P, against Lemma 2.

The same argument shows that i covers at most 2 elements. □

Lemma 4. *If the subposets*

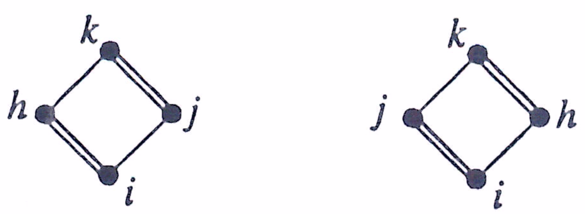
$Q =$

or $Q' =$

appear in the Hasse diagram of P, then they are not convex in P.

Proof. Suppose that Q is convex in P, then $L(Q)$ is plactic-closed (Corollary 1). We have $i < j$, $j > k$. If $i < k$, then $i < k < j$ and $ijk \sim jik \in L(Q)$ [resp. $i > k$, $k < i < j$, $ijk \sim ikj \in L(Q)$], a contradiction, because $L(Q) = \{ijk\}$. The case of Q' can be proven in the same way, using Knuth's relations. □

Corollary 2. *In the hypothesis of Lemma 4 there exists a unique $h \neq j$ with $i \prec_P h \prec_P k$ and in P we have the following configurations (corresponding respectively to Q and Q')*

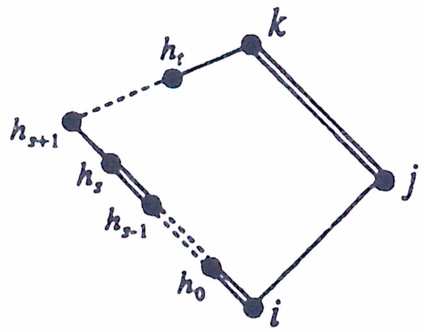


Proof. Existence of h: if Q appears in P, by Lemma 4, there exists a chain

$$\underline{h} = (i \prec_P h = h_0 \prec_P h_1 \prec_P \cdots \prec_P h_t \prec_P h_{t+1} = k)$$

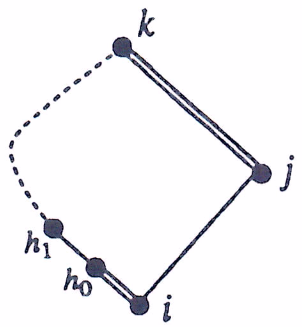
from i to k different than $i \prec_P j \prec_P k$. Observe that $h_1 \neq j$, because if $i \prec_P h_0 \prec_P h_1 = j$ then j does not cover i. Note also that by virtue of Lemma 2, $i > h = h_0$ since $i < j$. We want to show that $t = 0$.

For any such chain $\underline{h}$ define $l(\underline{h})$ to be the integer $0 \leq s \leq t$ such that $h_0 > h_1 > \cdots > h_s$ and $h_s < h_{s+1}$.



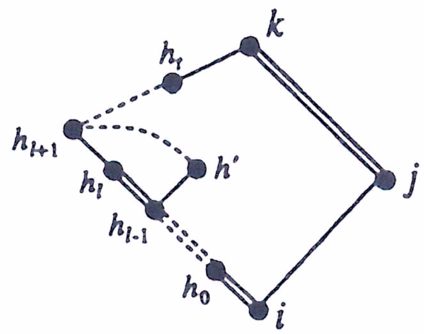
This number is well defined because we have $h_t < h_{t+1} = k$. Indeed, since $j > k = h_{t+1}$, we could not have also $h_t > h_{t+1} = k$, as this would lead to one of the configurations forbidden by Lemma 2.

Let $\underline{h}$ be a chain such that $l(\underline{h}) = l$ is minimal: we claim that $l = 0$. If so, the subposet $i \prec_P h_0 \prec_P h_1$ satisfies the hypothesis of Lemma 4 and then it is not convex in P:



So there exists $j' \neq h_0$ with $i \prec_P j' \leq_P h_1$ and by Lemma 3, $j' = j$. This implies $j \leq_P h_1 \leq_P k$, but k covers j: we deduce $h_1 = k$ and $t = 0$, as we wanted.

Assume $l > 0$. In this case the chain $h_{l-1} \prec_P h_l \prec_P h_{l+1}$, by Lemma 4, is not convex in P, because $h_{l-1} > h < h_{l+1}$: let $h' \in P$ be such that $h_{l-1} \prec_P h' \prec_P \cdots h_{l+1}$. Again, since $h_{l-1} > h_l$, we have, by Lemma 2, $h_{l-1} < h'$.



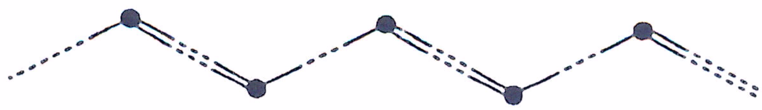
Now define a new chain $\underline{h}'$ by replacing in $\underline{h}$ the subchain $h_{l-1} \prec_P h_l \prec_P h_{l+1}$ by $h_{l-1} \prec_P h' \cdots \prec_P h_{l+1}$. We have $l(\underline{h}') = l - 1 < l(\underline{h})$, against minimality of l.

Uniqueness of h: any other element $h' \neq h$ with $i \prec_P h' \prec_P k$, would lead to a contradiction with Lemma 3. □

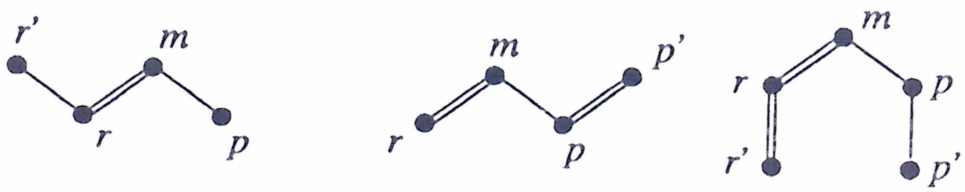
Definition 1. *A stair-like path* is a sequence $(p_1, p_2, \ldots, p_t)$ of points of P, possibly empty, connected in the Hasse diagram and such that

$$p_1 < p_2 < \cdots < p_t.$$

Note that a stair-like path has the form



Lemma 5. *If one of the three following configurations are subposets of the Hasse diagram of P*



then we cannot find in P a stair-like path joining r' and p, r and p', r' and p' respectively.

Proof. It is easily seen that $r' > p$ [resp. $r > p'$, $r' > p'$]. If a stair-like path exists, then we would have $r' < p$ [resp. $r < p'$, $r' < p'$], a contradiction. □

We are now ready to reconstruct the poset P in the discrete plane $\mathbf{N} \times \mathbf{N}$ with its natural partial order

$$(x, y) \leq (x', y') \Leftrightarrow x \leq x' \quad \text{and} \quad y \leq y'.$$

Then the covering relation $\prec$ in $\mathbf{N} \times \mathbf{N}$ is given by

$$(x, y) \prec (x', y') \Leftrightarrow x' = x,\ y' = y + 1 \quad \text{or} \quad x' = x + 1,\ y' = y.$$

Definition 2. We call a *skew tableau* of weight n any finite convex subposet T of $\mathbf{N} \times \mathbf{N}$, labelled with the elements of P, such that the labelling, with respect to the total order on P, is row-increasing reading from left to right and column-decreasing reading from the bottom to the top.

Definition 3. In $\mathbf{N} \times \mathbf{N}$ a *path* from (x, y) to (x', y') is a sequence of points

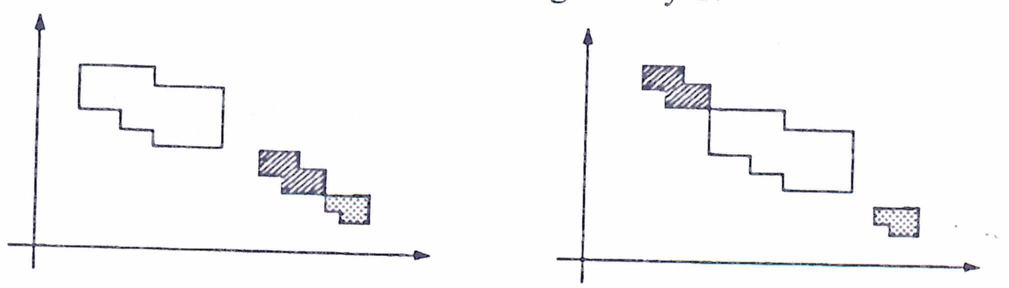
$$((x, y) = (x_1, y_1); (x_2, y_2); \ldots; (x_k, y_k) = (x', y')$$

such that, for $i = 1, \ldots, k - 1$ either $(x_i, y_i) \prec (x_{i+1}, y_{i+1})$ or $(x_{i+1}, y_{i+1}) \prec (x_i, y_i)$.

Definition 4. Let T be a subposet of $\mathbf{N} \times \mathbf{N}$. Two points (x, y) and (x', y') are *connected* in T if there exists a path joining them whose elements are all in T.

Remark 2. In a skew tableau T two comparable points (x, y) and (x', y') are connected in T: indeed any saturated chain between (x, y) and (x', y') is a path whose points are all in T, by convexity of T.

Remark 3. In the discrete plane, "shifting" the connected components of a skew tableau T does not change the partial order given by T.



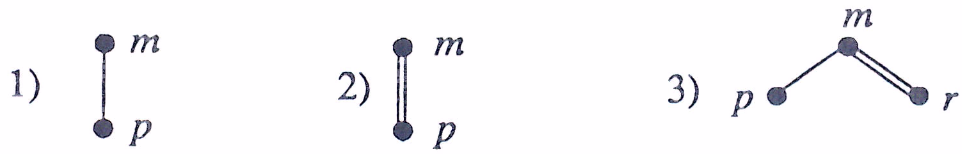
So the Theorem will take the following form: if $L(P)$ is plactic-closed, then there exists an embedding e: $P \to \mathbf{N} \times \mathbf{N}$ which respects the order $\leq_P$ and such that $e(P)$, with the labelling ω defined as $\omega(e(p)) = p$, is a skew tableau.

Proof of Theorem 1. By induction on $|P|$. If $|P| \leq 2$, then the statement is true as we can have the following possibilities for P

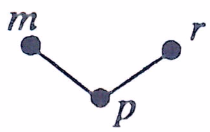
which correspond respectively to the diagrams of shape (1), (2) and (11). Now suppose $|P| \geq 3$, and let m be a maximal element of $(P, \leq_P)$. It is easily seen that $Q = P \backslash \{m\}$ is still a convex subposet of P, hence (by Corollary 1) plactic-closed. Then by the induction hypothesis, there is an embedding e: $Q \to \mathbf{N} \times \mathbf{N}$ and $e(Q)$ is a skew tableau, with the labelling ω defined by: $\omega(e(q)) = q$.

We wish to extend e "correctly" to all of P, i.e., to glue m to the skew shape given by $e(Q)$ in a way such that the result is still a skew shape.

As shown in Lemma 3, m can cover at most two elements, so that there are three possible cases in P:

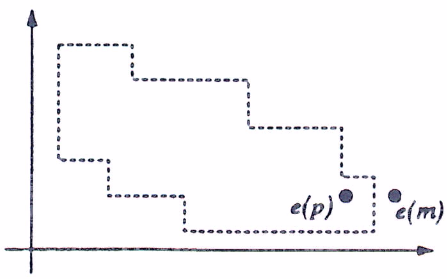


Case 1. Observe that there is nothing at the right of and on the same line as $e(p)$ in $e(Q)$, otherwise there exists $r \neq m$ such that



is a subposet of P which contradicts Lemma 2. Hence we can extend e to P by

$$e(m) = e(p) + (1, 0) = (x_p + 1, y_p)$$

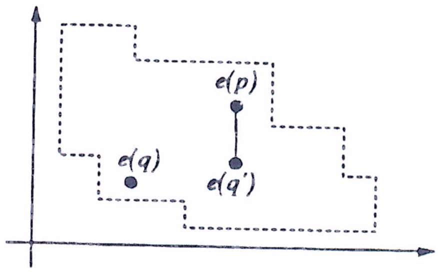


where we write (x_p, y_p) for the coordinates of $e(p)$ in $\mathbf{N} \times \mathbf{N}$. Note that e respects the covering relations in P, hence e respects $\leq_P$. Thus to show that $e(P)$ is a skew tableau, we need only to verify the convexity.

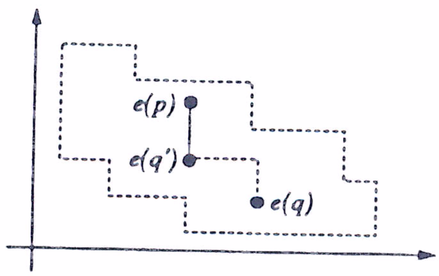
Claim. For any $q \in Q$ such that $e(q)$ is in the same connected component as $e(p)$, the inequality $y_q \geq y_p$ holds: if not, there exists an element $e(q) = (x_q, y_q)$ with $y_q < y_p$, that is $e(q)$ is below $e(p)$. We can always choose $e(q) = (x_p, y_p - 1)$, in which case the diagram

is a subposet of P, and by Corollary 2 there would be another point covered by m, against assumption. Indeed, if $e(q)$ is to the left of $e(p)$, i.e., $x_q < x_p$, then

$$e(q) \leq (x_p, y_p - 1) \leq e(p)$$



and by convexity of $e(Q)$ there exists $q' \in Q$ with $e(q') = (x_p, y_p - 1)$. If $e(q)$ is to the right of $e(p)$, i.e., $x_q > x_p$, and since $e(p)$ and $e(q)$ are in the same connected component, there exists a path in $e(Q)$ joining them, and in particular there is $q' \in Q$ with $e(q') = (x_p, y_p - 1)$.



In fact, we can assume that the inequality $y_q \geq y_p$ holds as well when $e(q)$ is not connected to $e(p)$: we can define another embedding e', shifting in the plane the connected component of $e(q)$ above that of $e(p)$ (Remark 3), and the induction hypothesis on $e'(Q)$ will still apply.

Suppose now that $e(P)$ is not convex: then there exists a point $(x, y) \notin e(P)$ and $q \in P$ with $e(q) \leq (x, y) \leq e(m)$ (the case $e(m) \leq (x, y) \leq e(q)$ has not to be considered, as it contradicts the maximality of m). Remark that (x, y) cannot be comparable with $e(p)$: indeed, if $e(p) < (x, y)$, then

$$e(p) < (x, y) \leq e(m),$$

but in $\mathbb{N} \times \mathbb{N}$, $e(p)$ is covered by $e(m)$; if $(x, y) < e(p)$, then

$$e(q) \leq (x, y) < e(p)$$

against the convexity of $e(Q)$, which was assumed by induction.

Now we have $y_q \leq y \leq y_m = y_p$ and also, by the claim, we have the inequality $y_q \geq y_p$: this implies $y_q = y_p = y$, which means that (x, y) is comparable with (x_p, y_p), a contradiction.

Case 2. Analoguous to case 1.