Characteristic properties and recognition of graphs in which geodesic and monophonic convexities are equivalent

Francesco M. Malvestuto, Mauro Mezzini and Marina Moscarini

Department of Computer Science
Sapienza University of Rome, Italy

October 17, 2012

Abstract

Let G be a connected graph. A subset X of $V(G)$ is g-convex (m-convex) if it contains all vertices on shortest (induced) paths between vertices in X. We state characteristic properties of graphs in which every g-convex set is m-convex, based on which we show that such graphs can be recognized in polynomial time. Moreover, we state a new convexity-theoretic characterization of Ptolemaic graphs.

Keywords
Geodesic convexity
Monophonic convexity
Minimal vertex separators
γ-acyclic hypergraphs
Ptolemaic graphs

1 Introduction

A convexity space on a connected graph G is any set of subsets of $V(G)$ which contains the empty set, the singletons and $V(G)$, and is closed under set intersection. Several notions of convexity were introduced using different path types; for example, shortest paths (geodesics), induced (or minimal or chordless) paths and generic paths were used to define geodesic convexity (or g-convexity) [8] [10] [21], monophonic convexity (or m-convexity) [6] [8], and all-paths convexity (or ap-convexity) [20] [3], respectively. It is not difficult to prove that m-convexity and ap-convexity are equivalent in G if and only if G is a tree [17]. On the other hand, very little is known about those graphs in which g-convexity and m-convexity are equivalent. Of course, they are equivalent in distance-hereditary graphs, since there every induced path is a shortest path. The only remarkable result was stated by Farber and Jamison [8], who proved that, within the class of connected chordal graphs, g-convexity and m-convexity are equivalent in G if and only if G is Ptolemaic (i.e., chordal and distance-hereditary). From the solution to the equivalence problem above one could learn something more about certain parameters of a graph such as its m-hull number (mhn), its m-number (mn), its g-number (gn), and its g-hull number (mhn) [11], for which no general relationship is known apart from the following inequalities $mhn \leq mn \leq gn \leq ghn$ [11]. The difficulty in finding a characterization (e.g., by forbidden induced subgraphs) of graphs in which g-convexity and m-convexity are equivalent is due to the fact that such graphs can have any graph as induced subgraph. To see it, let G_0 be any nonempty graph and let G be the graph obtained from G_0 by adding two (nonadjacent) vertices u and v, which are made adjacent to every vertex of G_0. Then, a nonempty subset of $V(G)$ is g-convex if and only if it is either a clique of G or $V(G)$ itself. Therefore, since every m-convex set of G is also g-convex and the cliques of G are all m-convex sets, one has that a subset of $V(G)$ is g-convex if and only if it is m-convex.

In this paper, we make use of prime components (i.e., maximal subgraphs containing no clique separators) of a graph in order to characterize those graphs G in which g-convexity and m-convexity are equivalent and we give both a "local" property (g-convexity and m-convexity are equivalent in every prime component of G) and "superstructural" properties, which state how prime components of G are linked to one another. Moreover, based on these properties, we provide a polynomial-time algorithm to recognize such graphs. Finally, we state a stronger result than the above mentioned result by Farber and Jamison by proving that, within the class of connected bridged graphs, g-convexity and m-convexity are equivalent in G if and only if G is Ptolemaic.

The paper is organized as follows. Section 2 contains basic definitions and preliminary results on minimal vertex separators, on α- and γ-acyclic hypergraps, and on g- and m-convexities. In Section 3 we give some convexity-theoretic properties of prime components of a graph. Section 4 contains three characterizations of graphs in which g-convexity and m-convexity are equivalent. In Section 5 we show that graphs in which geodesic and monophonic convexities are equivalent can be recognized in $O(n^4 m)$ time, where n is the number of vertices and m the number of edges. Finally, in Section 6 we provide a new convexity-theoretic characterization of Ptolemaic graphs.

2 Basic definitions and preliminary results

In what follows G will be a finite, connected, undirected, loopless and simple graph.

A sequence $(v_1, \ldots, v_k, v_{k+1})$ where the v_i, $1 \leq i \leq k$, are distinct vertices of G and v_i and v_{i+1}, $1 \leq i \leq k$, are adjacent, is a v_1-v_{k+1} *path* of length k if v_{k+1} is different from the other v_i's, and is a *cycle* of length k if $k > 2$

and $v_1 = v_{k+1}$. A *subpath* of a path $(v_1, \dots, v_k, v_{k+1})$ is any path of the type $(v_{i_1}, \dots, v_{i_h})$ with $i_1 < \dots < i_h$. Let u and v be two vertices; a u-v *geodesic* is a u-v path of minimum length; the *distance*, $d(u, v)$, of u and v is the length of a u-v geodesic.

2.1 Minimal vertex separators

Let S be a proper subset of $V(G)$; the *neighborhood* of S in G, denoted by $N(S)$, is the set of vertices in $V(G) - S$ that are adjacent to some vertex in S; by $G - S$ we denote the subgraph of G induced by $V(G) - S$. An S-*component* of G is a connected component K of $G - S$ such that $N(V(K)) = S$.

Two vertices of G are *separated* by S if they belong to distinct connected components of $G - S$. S is a *minimal separator* for two vertices u and v if u and v are separated by S and by no proper subset of S; S is a *minimal vertex separator* of G if there exist two vertices for which S is a minimal separator. It is well-known that the minimal vertex separators of a chordal graph are all cliques. We now recall and state some properties of minimal vertex separators.

Fact 2.1. *[13] Let S be a minimal separator for u and v. Let K and K' be the connected components of $G - S$ containing u and v, respectively. Every vertex in S is adjacent to a vertex of K and to a vertex of K'.*

Lemma 2.1. *S is a minimal separator for u and v if and only if u and v belong to two distinct S-components of G.*

Proof. *(Only if)* Let K and K' be the connected components of $G-S$ containing u and v, respectively. By Fact 2.1, $N(V(K)) = N(V(K')) = S$.
(If) Let K and K' be the S-components of $G-S$ containing u and v, respectively. Since $N(V(K)) = N(V(K')) = S$, no proper subset of S separates u and v. □

Corollary 2.1. *For every minimal vertex separator S there exist at least two S-components of G.*

Lemma 2.2. *Let S be a minimal vertex separator of G. Every vertex in S is on an induced path between every pair of vertices for which S is a minimal separator.*

Proof. Let us suppose, by contradiction, that there exist two vertices u and v for which S is a minimal separator and there exists a vertex w in S that lies on no induced u-v path. Since S is a minimal separator for u and v, $S' = S - \{w\}$ does not separate u and v. Therefore, $\{w\}$ is a minimal vertex separator of $G - S'$ and, hence, w lies on every (induced) u-v path in $G - S'$. Since every (induced) u-v path in $G - S'$ is an (induced) u-v path in G, a contradiction arises. □

2.2 Hypergraph acyclicity

A minimal vertex separator and a clique of a hypergraph are defined in a similar way as in a graph. Moreover, a *partial edge* is any nonempty subset of some edge.

Fagin [7] introduced four notions of hypergraph acyclicity which prove to be relevant in the study of convexity in hypergaphs [18] [16] [17]. We now recall the definitions of α-acyclic and γ-acyclic hypergraphs.

A hypergraph H is α-*acyclic* if it is the clique hypergraph of a chordal graph, where the *clique hypergraph* of a graph G is the hypergraph whose edges are exactly the maximal cliques of G.

Since every minimal vertex separator of a chordal graph is a clique, one has that every minimal vertex separator of an α-acyclic hypergraph is a partial edge.

Connected α-acyclic hypergraphs can be represented by trees (e.g., see [2]). We shall make use of another tree representation (see [19]) whose definition is now recalled. Let H be a connected α-acyclic hypergraph and let M be the set of minimal vertex separators of H. A *connection tree* (also called an "edge-divider" tree [1]) for H is a tree T with vertex set $H \cup M$, such that:

1. each edge of T has one end vertex in H and the other in M, and
2. for every two vertices X and Y of T, the set $X \cap Y$ is a subset of each vertex along the unique path joining X and Y in T.

Example 2.1 Let H be the α-acyclic hypergraph in Figure 1; a connection tree T for H is shown in Figure 2.

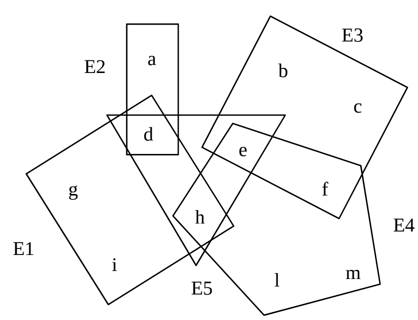

Figure 1: An α-acyclic hypergraph

In what follows, the vertices of T that are in H (or in M) are called H-vertices (M-vertices, respectively) of T. Let $S \subset V(H)$. By T_S we denote the forest obtained from T by deleting the vertices that are subsets of S. If T' is a connected component of T_S, by $V(T')$ we denote the union of the vertices (viewed as sets) of T'.

Lemma 2.3. *Let H be a connected α-acyclic hypergraph. Let $S \subset V(H)$ and let u and v be two distinct vertices in $V(H) - S$. The vertices u and v are*

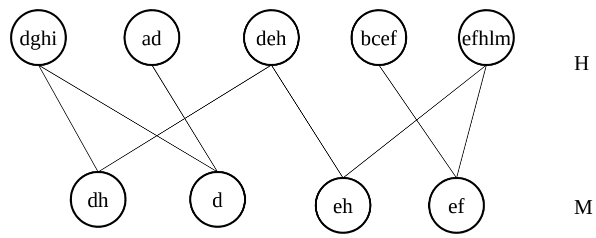

Figure 2: A connection tree T for the hypergraph in Figure 1

connected in $H - S$ if and only if they belong to two H-vertices of T that are connected in T_S.

Proof. If u and v are adjacent in H, then the statement is trivially true. Therefore, without loss of generality, we can assume that u and v are not adjacent in H so that they belong to two distinct H-vertices of T.
(*If*) Let E and E' be two H-vertices of T which are connected in T_S and contain u and v, respectively. Consider the path $(E_1 = E, S_1, E_2, \ldots, S_{k-1}, E_k = E')$ from E to E' in T_S. First of all, observe that $S_h - S \neq \emptyset$, for all h. Let $i = max(h|u \in E_h)$ and let $j = min(h|v \in E_h)$. Since u and v are not adjacent in H, one has $i < j$. Let $(u, x_i, \ldots, x_{j-1}, v)$ be a sequence of vertices obtained by taking for each h, $i \leq h \leq j-1$, x_h in $S_h - S$. Note that $x_h \notin S$ for all h and that every two consecutive vertices in the sequence are adjacent in H. Then, from the vertex sequence $(u, x_i, , x_{j-1}, v)$ we can easily obtain a u-v path in H which does not pass through S, which proves that u and v are connected in $H - S$.

(*Only if*) Let $(x_1, \ldots, x_{k-1}, x_k)$ be a u-v path in H that does not pass through S. Let $i_1 = max(h > 1|x_h$ is adjacent to $x_1)$ and let E_1 be an edge of H that contains x_1 and x_{i_1}; note that $E_1 - S \supseteq \{x_1, x_{i_1}\}$. Analogously, let $i_2 = max(h > i_1|x_h$ is adjacent to $x_{i_1})$ and let E_2 be an edge of H that contains x_{i_1} and x_{i_2}; so, one has that $E_2 - S \supseteq \{x_{i_1}, x_{i_2}\}$ and that $x_{i_1} \in E_1 \cap E_2$. It follows that E_1 and E_2 are H-vertices of T_S and x_{i_1} belongs to each vertex in the path in T between E_1 and E_2, which implies that E_1 and E_2 are connected in T_S. Repeating this argument, we obtain a sequence $E_1, E_2, \ldots, E_q$ of edges of H such that $u \in E_1$, $v \in E_q$ and, for each j, $1 \leq j \leq q-1$, E_j and E_{j+1} are connected in T_S. It follows that E_1 and E_q are connected in T_S, which proves the statement. □

From Lemma 2.3 it follows that the connected components of $H - S$ correspond one-to-one to the connected components of T_S. More precisely, if H' is a connected component of $H - S$, then there exists one connected component T' of T_S such that $V(T') - S = V(H')$, and if T' is a connected component of T_S, then there exists one connected component H' of $H - S$ such that $V(H') = V(T') - S$. Moreover, if H' is a connected component of $H - S$ and T' is the corresponding connected component of T_S, then one has $N(V(H')) = V(T') \cap S$. Finally, H' is an S-component of H if and only if $S \subset V(T')$.

Fact 2.2. *Let H be a connected α-acyclic hypergraph, let T be a connection tree for H and let S be a minimal vertex separator of H. A connected component of $H - S$ is an S-component if and only if the corresponding connected component of T_S contains an H-vertex that in T is adjacent to the M-vertex S.*

Example 2.1 (continued) Consider the minimal vertex separator $S = \{d, h\}$ of H. The vertices in M that are subsets of S are $\{d, h\}$ and $\{d\}$. The forest T_S is shown in Figure 3. By Fact 2.2, there are two S-components of H with vertex sets $\{g, i\}$ and $\{b, c, e, f, l, m\}$, respectively.

We now introduce the notion of γ-acyclicity (see Definition 4 of γ-cyclicity in [7]).

A hypergraph is γ-*acyclic* if, for every pair of nondisjoint edges E and E', $E \cap E'$ separates every vertex in $E - E'$ from every vertex in $E' - E$. For example, the hypergraph in Figure 1 is not γ-acyclic because the vertices d and e are not separated by $E_1 \cap E_4$.

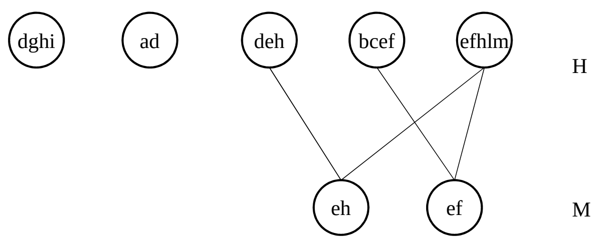

Figure 3: The forest T_S for $S = \{d, h\}$

2.3 g- and m-convexities

In this subsection we recall the notions of g-convexity and m-convexity.

Let u and v be two vertices of G. By $I_g(u, v)$ we denote the set of vertices that lie on any u-v geodesic. Let X be a subset of $V(G)$; by $I_g(X)$ we denote the set $\bigcup_{u,v \in X} I_g(u, v)$ with the convention $I_g(\emptyset) = \emptyset$; X is g-*convex* if $I_g(X) = X$ and the g-convex hull $\langle X \rangle_g$ is the minimal g-convex set of G containing X. By $g(G)$ we denote the set of g-convex sets of G.

Let u and v be two vertices of G. By $I_m(u, v)$ we denote the set of vertices that lie on any induced u-v path. Let X be a subset of $V(G)$; by $I_m(X)$ we denote the set $\bigcup_{u,v \in X} I_m(u, v)$ with the convention $I_m(\emptyset) = \emptyset$; X is m-*convex* if $I_m(X) = X$ and the m-convex hull $\langle X \rangle_m$ is the minimal m-convex set of G containing X. By $m(G)$ we denote the set of m-convex sets of G.

3 Prime components of a graph

A graph is *clique separable* if it contains two vertices separated by a clique, and is *prime* otherwise. A *prime component* (also called "maximal prime subgraph" [15]) of a graph G is a maximal induced subgraph of G that is prime.

A *minimal clique separator* for two vertices u and v is a clique that is a minimal separator for u and v; a *minimal vertex clique separator* of G is a clique that is a minimal vertex separator of G.

The *prime hypergraph* of G is the hypergraph whose edges are the vertex sets of the prime components of G.

Example 3.1 Let G be the graph in Figure 4. The prime hypergraph of G coincides with the hypergraph H shown in Figure 1. A connection tree T for H was shown in Figure 2.

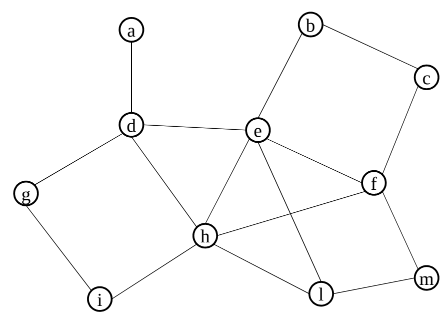

Figure 4: The graph in Example 3.1

Fact 3.1. *Let H be the prime hypergraph of a graph G.*

- *The minimal vertex separators of H are exactly the minimal vertex clique separators of G.*
- *If C is a minimal vertex clique separator of G then, for every C-component of G, there exists a C-component of H with the same vertex set, and vice versa.*

By Facts 2.2 and 3.1 one has the following.

Fact 3.2. *Let C be a minimal vertex clique separator of a graph G. For every C-component K of G there exists a prime component P such that $V(P) - C \subseteq V(K)$ and $V(P) \supseteq C$.*

Example 3.1 (continued) Consider the minimal vertex clique separator $C = \{d, h\}$ of G. By Fact 3.1, C is a minimal vertex clique separator of G and there are exactly two C-components K and K' of G with vertex sets $\{g, i\}$ and $\{b, c, e, f, l, m\}$, respectively (see Example 2.1). We have that E_1 is the vertex set of a prime component of G such that $E_1 - C \subseteq V(K)$ and $E_1 \supseteq C$, and E_5 is the vertex set of a prime component of G such that $E_5 - C \subseteq V(K')$ and $E_5 \supseteq C$.

Fact 3.3. *[15] The prime hypergraph of a graph G is α-acyclic.*

In Section 4 we will show that if $g(G) = m(G)$ then the prime hypergraph of G is γ-acyclic.

We now recall a result on m-convex hulls involving prime components and minimal vertex clique separators. Let X be a subset of $V(G)$. In the following

- by X' we denote the union of X with all minimal clique separators for pairs of vertices in X, and
- by X'' the union of the vertex sets of prime components P of G such that $X' \cap V(P)$ is neither the empty set nor a clique.

Example 3.1 (continued) For $X = \{a, e, f, g\}$, we have $X' = X \cup \{d\} \cup \{d, h\} \cup \{e, h\}$ and $X'' = \{d, g, h, i\}$.

In [18] (see Theorem 8) it is proven that:

Lemma 3.1. *For every subset X of $V(G)$, one has $\langle X \rangle_m = X' \cup X''$.*

As a consequence of Lemma 3.1 we obtain the following characterization of m-convex sets in a prime graph.

Corollary 3.1. *Let G be a prime graph. $m(G)$ consists of the empty set, the cliques and $V(G)$.*

By Lemma 3.1, if P is a prime component of G and $X \subseteq V(P)$, then $\langle X \rangle_m \subseteq V(P)$. On the other hand, since for every subset X of $V(G)$, $\langle X \rangle_g \subseteq \langle X \rangle_m$, from Lemma 3.1 it also follows:

Corollary 3.2. *Let P be a prime component of G. A subset of $V(P)$ is g-convex (m-convex) in G if and only if it is g-convex (m-convex) in P.*

4 Characteristic properties

In this section we state some characteristic properties of a connected graph G in which g-convexity and m-convexity are equivalent. To this aim we need the following technical lemma.

Lemma 4.1. *Let C be a minimal vertex clique separator of G and let K and K' be two C-components of G. If $g(G) = m(G)$ then, for every pair of vertices $u \in N(C) \cap V(K)$ and $v \in N(C) \cap V(K')$, $d(u, v) = 2$ and $\langle\{u, v\}\rangle_g = I_g(u, v)$.*

Proof. Let u be a vertex in $N(C) \cap V(K)$ and v a vertex in $N(C) \cap V(K')$. Observe that $d(u, v) > 1$, since u and v are separated by C, and $d(u, v) < 4$, since C is a clique. So, either $d(u, v) = 2$ or $d(u, v) = 3$. We will show that:

(1) if $d(u, v) = 2$ then $\langle\{u, v\}\rangle_g = I_g(u, v)$, and

(2) $d(u, v) \neq 3$.

(1) If $d(u, v) = 2$ then every vertex in $I_g(u, v)$ distinct from both u and v belongs to C (otherwise C would not separate u and v); therefore, since C is a clique, $\langle\{u, v\}\rangle_g = I_g(u, v)$.

(2) Suppose, by contradiction, that $d(u, v) = 3$. Since $u \in N(C) \cap V(K)$, $v \in N(C) \cap V(K')$ and C is a clique, there exists a u-v geodesic (u, y, z, v) such that both y and z are in C.

By Fact 2.1, there exists a vertex x in K adjacent to z. Consider the two vertices x and v; we have that $x \in N(C) \cap V(K)$, $v \in N(C) \cap V(K')$ and $d(x, v) = 2$. Therefore, by (1) we have that $\langle\{x, v\}\rangle_g = I_g(x, v)$. Since $d(u, v) = 3$, y cannot be adjacent to v and, hence, y cannot belong to $\langle\{x, v\}\rangle_g = I_g(x, v)$. On the other hand, by Lemma 2.2, y belongs to $\langle\{x, v\}\rangle_m$. Therefore, $\langle\{x, v\}\rangle_g \neq \langle\{x, v\}\rangle_m$ and a contradiction arises. □

The next result provides a characterization of a connected graph G in which g-convexity and m-convexity are equivalent, which involves the following property:

(p1) For every minimal vertex clique separator C of G and for every pair of vertices u and v of two distinct C-components, every vertex in C is on a u-v geodesic.

Theorem 4.1. *$g(G) = m(G)$ if and only if:*

1. *$g(P) = m(P)$ for every prime component P of G, and*
2. *G has property (p1).*

Proof. (*Only if*) *Proof of (1).* By Corollary 3.2.

Proof of (2). Suppose, by contradiction, that there exist a minimal vertex clique separator C of G, two C-components K and K' of G, two vertices $u \in V(K)$ and $v \in V(K')$ and a vertex w in C that is on no u-v geodesic.

Let p be a u-v geodesic. Let x be the last vertex on p belonging to $V(K)$ and let y be the first vertex on p belonging to $V(K')$. The x-y subpath p' of p is an x-y geodesic. By Lemma 2.1, C is a minimal clique separator for x and y so that, by Lemma 2.2, $w \in I_m(x, y) \subseteq \langle\{x, y\}\rangle_m$. We now prove that $w \notin \langle\{x, y\}\rangle_g$, which contradicts the hypothesis $g(G) = m(G)$. By Lemma 4.1, $w \in \langle\{x, y\}\rangle_g$ if and only if $w \in I_g(x, y)$. If w were in $I_g(x, y)$, then there would exist an x-y geodesic p'' such that w lies on p'', but then w would be on the u-v geodesic which is

obtained from p by substituting p' with p''. Since w is on no u-v geodesic, $w \notin I_g(x,y)$ and, hence, $w \notin \langle\{x,y\}\rangle_g$.
(*If*) Let X be any nonempty subset of $V(G)$. If there exists a prime component P of G such that $X \subseteq V(P)$ then, by Corollary 3.2 and condition (1), $\langle X\rangle_g = \langle X\rangle_m$. Assume that this is not the case, that is, X contains two vertices that are separated by a clique. Since $\langle X\rangle_g \subseteq \langle X\rangle_m$, in order to prove that $\langle X\rangle_g = \langle X\rangle_m$ we need to prove, by Lemma 3.1, that:

(i) $\langle X\rangle_g \supseteq X'$
(ii) $\langle X\rangle_g \supseteq X''$.

Proof of (i). Let C be any minimal clique separator for a pair of vertices in X. By Lemma 2.1 these two vertices belong to two distinct C-components so that, by condition (2), C is a subset of $\langle X\rangle_g$. Therefore, $\langle X\rangle_g \supseteq X'$.
Proof of (ii). If $X'' = \emptyset$ then trivially $X'' \subseteq \langle X\rangle_g$. Otherwise, let P be any prime component of G such that $X' \cap V(P)$ is neither the empty set nor a clique. Let u and v be two nonadjacent vertices in $X' \cap V(P)$. By Corollaries 3.1 and 3.2, $\langle\{u,v\}\rangle_m = V(P)$ so that, by condition (1), $\langle\{u,v\}\rangle_g = V(P)$. Finally, since $\{u,v\} \subseteq X'$ and $X' \subseteq \langle X\rangle_g$ (see above), one has $V(P) = \langle\{u,v\}\rangle_g \subseteq \langle X\rangle_g$. It follows that $X'' \subseteq \langle X\rangle_g$.

□

We shall provide two more characterizations of a graph in which g-convexity and m-convexity are equivalent (see Theorem 4.2 below). To this aim we relate property (p1) to the following:

(p2) For every minimal vertex clique separator C of G and for every C-component K of G and for every vertex $u \in V(K) \cap N(C)$, the set $C \cup \{u\}$ is a clique.

(p3) For every minimal vertex clique separator C of G and for every prime component P of G containing C and for every vertex $u \in V(P) \cap N(C)$, the set $C \cup \{u\}$ is a clique.

(p4) The prime hypergraph H of G is γ-acyclic.

Lemma 4.2. *The following conditions are equivalent:*
(i) G has property (p1)
(ii) G has property (p2)
(iii) G has properties (p3) and (p4).

Proof. (i) $\Rightarrow$ (ii). Let C be any minimal vertex clique separator of G, let K be any C-component of G and let u be any vertex in $V(K) \cap N(C)$. Let w be a vertex in C adjacent to u. By Corollary 2.1, there exists another C-component K' of G. Let v be a vertex of K' adjacent to w (such a vertex exists by Fact 2.1). Of course, (u,w,v) is a geodesic. By (p1) every vertex in C is on a u-v geodesic and, hence, $C \cup \{u\}$ is a clique which proves that G has property (p2).

(ii) $\Rightarrow$ (i). Let C be any minimal vertex clique separator of G, let K and K' be any two C-components of G and let $u \in V(K)$ and $v \in V(K')$. Let p be a u-v geodesic; let x be the last vertex on p belonging to $V(K)$ and let y be the first vertex on p belonging to $V(K')$. Since the vertex on p following x is in C and, analogously, the vertex on p preceding y is in C, by (p2), both x and y are adjacent to every vertex in C. Let p_1 be the u-x subpath of p and let p_2 be the y-v subpath of p. We have that for every vertex w in C the path p_1, w, p_2 is a u-v geodesic which proves that G has property (p1).

(ii) $\Rightarrow$ (iii).
Proof of (p3). Let C be any minimal vertex clique separator of G and let P be any prime component of G containing C. Let K be the C-component of G containing $V(P)-C$. Since every vertex u in $V(P)\cap N(C)$ is in $V(K)\cap N(C)$, by (p2) the set $C\cup\{u\}$ is a clique.
Proof of (p4). Suppose, by contradiction, that the prime hypergraph H of G is not γ-acyclic. Then there exist two prime components P and P' of G such that $S = V(P)\cap V(P') \neq \emptyset$ and

(a) S separates no vertex in $V(P)-S$ from no vertex in $V(P')-S$.

By the very definition of a prime component of a graph, there exists a clique separator $C \subseteq V(P)$ such that:

(b) C is a minimal clique separator for every pair of vertices, one in $V(P)-C$ and the other in $V(P')-C$.

Analogously, there exists a clique separator $C' \subseteq V(P')$ such that:

(b') C' is a minimal clique separator for every pair of vertices, one in $V(P)-C'$ and the other in $V(P')-C'$.

Since $S = V(P)\cap V(P')$, we have that $S \subset C$ (otherwise, C would not separate any pair of vertices, one in $V(P)-C$ and the other in $V(P')-C$); analogously, $S \subset C'$. From (a) it follows that:

(c) S separates no vertex in $C-C'$ from no vertex in $C'-C$.

Let $v \in C'-C$, $s \in S$ and $x \in V(P)-C$. There exists an s-x path p in P that does not pass through $C-S$, for, otherwise, s and x would be separated by a clique, which contradicts that P is prime. Let u be the first vertex on p that is not in S. By (b), C separates u and v, so that u and v are not adjacent. Let K be the connected component of $G-C'$ containing u; by (c), we have that $C-C' \subseteq V(K)$. Let us show that K is a C'-component. Suppose, by contradiction, that $N(V(K)) \subset C'$; then, $N(V(K))$ would separate every pair of vertices, one in $V(P)-C'$ and the other in $V(P')-C'$ contradicting (b'). Therefore, K is a C'-component. Since u and v are not adjacent, $C'\cup\{u\}$ is not a clique and a contradiction with (p2) arises.

(iii) $\Rightarrow$ (ii). Let C be any minimal vertex clique separator and let K be any C-component of G. Since the prime hypergraph H of G is γ-acyclic, a nonempty subset of $V(G)$ is a minimal vertex clique separator if and only if it is the intersection of two distinct prime components. By Corollary 2.1 and Fact 3.2 there exist two prime components P and P' such that $C = V(P)\cap V(P')$, $V(P)-C \subseteq V(K)$, $V(P) \supseteq C$ and $V(P')\cap V(K) = \emptyset$. Let u be a vertex in $V(K)\cap N(C)$. If $u \in V(P)$, then, by (p3), the set $C\cup\{u\}$ is a clique. Otherwise, let v be a vertex in C adjacent to u and let Q be a prime component containing both u and v. If $C-V(Q)$ were not empty, then the nonempty set $S = V(Q)\cap V(P')$ would not separate u from any vertex in $V(P')-S$, which contradicts (p4). Therefore, $V(Q) \supseteq C$ so that, by (p3), the set $C \cup \{u\}$ is a clique. □

From Theorem 4.1 and Lemma 4.2 it follows that:

Theorem 4.2. *The following statements are equivalent:*

- $g(G) = m(G)$
- $g(P) = m(P)$ *for every prime component* P *of* G*, and* G *has property (p1)*
- $g(P) = m(P)$ *for every prime component* P *of* G*, and* G *has property (p2)*
- $g(P) = m(P)$ *for every prime component* P *of* G*, and* G *has properties (p3) and (p4).*

5 Recognition

In this section we show that graphs in which geodesic and monophonic convexities are equivalent can be recognized in $O(n^4m)$ time (where n is the number of vertices and m the number of edges) using the following characterization given in Theorem 4.1: $g(G) = m(G)$ if and only if

1. $g(P) = m(P)$ for every prime component P of G, and
2. for every minimal vertex clique separator C of G and for every pair of vertices u and v of two distinct C-components, every vertex in C is on a u-v geodesic.

5.1 Testing condition (1)

The prime components of G and its minimal vertex clique separators can be computed using the $O(nm)$ decomposition algorithm given in [22] and modified by [15]. As noted by Tarjan [22], the number of prime components of G is at most $n-1$, for $n \geq 2$.

In [5] an $O(nm)$ algorithm to compute the g-convex hull of a given vertex set is given. By Corollary 3.1, in order to test $g(P) = m(P)$, for a given prime component P of G, it is sufficient to compute the g-convex hull of every pair of nonadjacent vertices and check that it is equal to $V(P)$.

Therefore, testing condition (1) requires $O(n^4m)$ time.

5.2 Testing condition (2)

It is well-known (for example, see [14]) that the number of minimal (clique) separators of a chordal graph G is at most $k-2$, where k is the number of its maximal cliques. Since the minimal vertex (clique) separators of the 2-section H_2 of the prime hypergraph H of G are exactly the minimal vertex clique separators of G and since the maximal cliques of H_2 are exactly the vertex sets of the prime components of G, which are at most $n-1$ (see above), the number of minimal vertex clique separators of G is at most $n-2$.

In order to test condition (2), for every minimal vertex clique separator C of G we have to perform the following two steps:

Step 1 find the C-components of G;
Step 2 for every pair of vertices u and v of two distinct C-components, compute $I_g(u,v)$ and check that $C \subseteq I_g(u,v)$.

Step 1 can be performed in $O(m)$ time during a traversal of G. Since computing $I_g(u,v)$ requires $O(m)$ time (by applying breadth first search) and checking the inclusion $C \subseteq I_g(u,v)$ requires $O(n)$ time, Step 2 can be performed in $O(n^2 m)$ time. Therefore, condition (2) can be tested in $O(n^3 m)$ time.

6 Ptolemaic graphs

Recall that a graph is Ptolemaic if it is connected, chordal and distance-hereditary [12]. Farber and Jamison [8] gave two convexity-theoretic characterizations of Ptolemaic graphs, one of which reads as follows:

Fact 6.1. *[8]. Let G be a connected graph. G is Ptolemaic if and only if G is chordal and $g(G) = m(G)$.*

We now state another characterization of Ptolemaic graphs stronger than Fact 6.1 by considering "bridged" graphs as defined by Farber [9].

A *bridge* of a cycle c in graph G is a geodesic in G joining two non consecutive vertices of c which is shorter than both of the paths in c joining those vertices. A graph G is *bridged* if every cycle of length at least 4 has a bridge. Of course, every chordal graph is a bridged graph.

Lemma 6.1. *If G is a bridged graph and $g(G) = m(G)$ then every prime component of G is a complete graph.*

Proof. Suppose, by contradiction, that there exists a prime component P of G that is not a complete graph and let u and v be two vertices in P with $d(u,v) = 2$. Since G is bridged, $N(u) \cap N(v)$ must be a clique, so that $I_g(u,v) = \langle\{u,v\}\rangle_g$. If $I_g(u,v) = V(P)$ then P is not prime (contradiction); if $I_g(u,v) \neq V(P)$ then, by Corollary 3.1, $\langle\{u,v\}\rangle_g \neq \langle\{u,v\}\rangle_m$, so that $g(G) \neq m(G)$ (contradiction). □

Lemma 6.2. *[4]. A connected graph is Ptolemaic if and only if its clique hypergraph is γ-acyclic.*

Theorem 6.1. *Let G be a connected graph. G is Ptolemaic if and only if G is a bridged graph and $g(G) = m(G)$.*

Proof. (Only if). If G is Ptolemaic then it is both chordal and distance-hereditary. Since every chordal graph is bridged and for every distance-hereditary graph g-convexity and m-convexity are equivalent, the statement trivially follows.

(If). If G is a bridged graph then, by Lemma 6.1, the prime hypergraph H of G coincides with the clique hypergraph of G. Moreover, if $g(G) = m(G)$ then, by Theorem 4.2, G has property (p4), that is, H is γ-acyclic. By Lemma 6.2, G is Ptolemaic. □

References

[1] J. H. Badsberg and F. M. Malvestuto. An implementation of the iterative proportional fitting procedure by propagation trees. *Computational statistics & data analysis*, 37:297–322, 2001.

[2] C. Beeri, R. Fagin, D. Maier, and M. Yannakakis. On the desirability of acyclic database schemes. *J. ACM*, 30(3):479–513, 1983.

[3] M. Changat, S. Klavzar, and H. M. Mulder. The all paths transit function on a graph. *Czechoslovak Mathematics J.*, 51 (126)(2):439–448, 2001.

[4] A. D'Atri and M. Moscarini. On hypergraph acyclicity and graph chordality. *Information Processing Letters*, 29(5):271–274, 1988.

[5] M. C. Dourado, J. G. Gimbel, J. Kratochvil, F. Protti, and J. L. Szwarcfiter. On the computation of the hull number of a graph. *Discrete Mathematics*, 309:5668–5674, 2009.

[6] P. Duchet. Convex sets in graphs II: minimal path convexity. *J. Combinatorial Theory Series B*, 44(3):307–316, 1988.

[7] R. Fagin. Degrees of acyclicity for hypergraphs and relational database schemes. *J. ACM*, 30(3):514–550, 1983.

[8] M. Farber and R. E. Jamison. Convexity in graphs and hypergraphs. *SIAM J. of Algebraic Discrete Mathematics*, 7(3):433–444, 1986.

[9] M. Farber. Bridged graphs and geodesic convexity. *Discrete Mathematics*, 66:249–257, 1987.

[10] F. Harary, E. Loukakis, and C. Tsouros. The geodetic number of a graph. *Math. Comput. Modelling*, 17:89–95, 1993.

[11] C. Hernando, M. Mora, I. M. Pelayo, and C. Seara. On geodesic and monophonic convexity. *XX European Workshop on Computational Geometry*, 2004.

[12] E. Howorka. A characterization of ptolemaic graphs. *Journal of Graph Theory*, 5:323–331, 1981.

[13] T. Kloks. *Treewidth, Computations and Approximations*, volume 842 of *Lecture Notes in Computer Science*. Springer, 1994.

[14] P. S. Kumar and C. V. Madhavan. Minimal vertex separators of chordal graphs. *Discrete Applied Mathematics*, 89:155–168, 1998.

[15] H.-G. Leimer. Optimal decomposition by clique separators. *Discrete Mathematics*, 113:99–123, 1993.

[16] F. M. Malvestuto, M. Mezzini, and M. Moscarini. Computing simple-path convex hulls in hypergraphs. *Information Processing Letters*, 111(5):231–234, 2011.

[17] F. M. Malvestuto, M. Mezzini, and M. Moscarini. Equivalence between hypergraph convexities. *ISRN Discrete Mathematics*, Article ID 806193:22 pages, 2011.

[18] F. M. Malvestuto. Canonical and monophonic convexities in hypergraphs. *Discrete Mathematics*, 309(13):4287–4298, 2009.

[19] F. M. Malvestuto. Tree and local computations in a cross–entropy minimization problem with marginal constraints. *Kybernetika*, 46(4):621–654, 2010.

[20] E. Sampathkumar. Convex sets in graphs. *Indian Journal of pure and applied Mathematics*, 15:1065–1071, 1984.

[21] V. P. Soltan. Metric convexity in graphs. *Studia Univ. Babes-Bolyai Math*, 36(4):3–43, 1991.

[22] R. E. Tarjan. Decomposition by clique separators. *Discrete Mathematics*, 55:221–232, 1985.