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A LINEAR ALGORITHM FOR FINDING THE INVARIANT EDGES OF AN EDGE-WEIGHTED GRAPH*

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Abstract. Given an edge-weighted graph where all weights are nonnegative reals, an edge reweighting is an assignment of nonnegative reals to edges such that, for each vertex, the sums of given and new weights assigned to the edges incident on the vertex do coincide. An edge is then said to be *invariant* if its weight is the same for any edge reweighting. We show that the set of invariant edges of an arbitrary edge-weighted graph can be determined in time linear in the size of the underlying graph. Moreover, an application to the security of statistical data is discussed.

Key words. linear algebra, graph algorithms, matroid theory

AMS subject classifications. 05B35, 05C50, 15A03, 62Q05, 68R10

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1. Introduction.

Let $G = (V, E)$ be a graph without isolated vertices (where self-loops and parallel edges may exist), and let $\boldsymbol{w} = (w(e))_{e\in E}$ be a vector of nonnegative reals. The pair $\Gamma = (G, \boldsymbol{w})$ is referred to as an *edge-weighted graph* EWG. Let $\mathbf{A}$ be the incidence matrix of G and let $\boldsymbol{b} = (b(v))_{v\in V}$ be the vector of nonnegative reals such that, for each vertex v of G, $b(v)$ equals the sum of the weights of the edges incident to v. Consider the following system of linear equations:

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}. \tag{1}$$

For every edge of G, let $L[x(e)]$ and $U[x(e)]$ denote, respectively, the minimum and the maximum of the variable $x(e)$ over the nonnegative solutions of equation system (1). An edge e of G is an *invariant edge* of Γ if $L[x(e)] = U[x(e)]$. Thus, an edge e of G is an invariant edge of Γ if and only if $x(e) = w(e)$ for every nonnegative solution of equation system (1). The following two examples show two EWGs which have all and no invariant edges, respectively.

EXAMPLE 1. *Consider the EWG $\Gamma = (G, \boldsymbol{w})$ shown in Figure 1, where α, β, and γ are any positive reals. By making use of standard algebraic methods, one finds there is no nonnegative solution of equation system (1) other than $\boldsymbol{w}$. Therefore, each edge of G is an invariant edge of Γ.* □

EXAMPLE 2. *Consider the EWG $\Gamma = (G, \boldsymbol{w})$ shown in Figure 2. The general expression of a nonnegative solution of equation system (1) is*

$$x(1,1) = 1 - 2\lambda, \qquad x(1,2) = x(1,3) = \lambda, \qquad x(2,3) = 1 - \lambda,$$

where the parameter λ ranges from 0 to 1/2. Therefore, one has

$$\begin{aligned} &L[x(1,1)] = 0, && U[x(1,1)] = 1,\\ &L[x(1,2)] = L[x(1,3)] = 0, && U[x(1,2)] = U[x(1,3)] = 1/2,\\ &L[x(2,3)] = 1/2, && U[x(2,3)] = 1, \end{aligned}$$

and hence, no edge of G is an invariant edge of Γ. □

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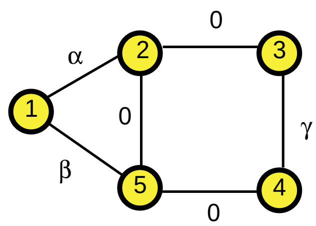

FIG. 1.

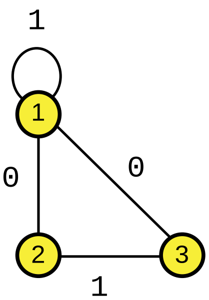

FIG. 2.

The problem addressed in this paper lies in finding the set of invariant edges of an arbitrary EWG. The following obvious fact allows us to limit our considerations to EWGs with underlying simple graphs (i.e., graphs without parallel edges).

Fact 1. Let $\Gamma = (G, \boldsymbol{w})$ be an EWG, where $G = (V, E)$ is a nonsimple graph. Let S be a set of two or more parallel edges and let e_0 be an arbitrarily chosen element of S. Let $G' = (V, E)$ be the graph with edge set $E' = (E - S) \cup \{e_0\}$. Consider the EWG $\Gamma' = (G', \boldsymbol{w}')$, where $\boldsymbol{w}'$ is defined as follows:

$$w'(e) = \begin{cases} w(e), & e \notin S, \\ \displaystyle\sum_{e' \in S} w(e'), & e = e_0. \end{cases}$$

Then, an edge not in S is an invariant edge of Γ if and only if it is an invariant edge of Γ', and an edge in S is an invariant edge of Γ if and only if $w'(e_0) = 0$ and e_0 is an invariant edge of Γ'.

The problem of finding the set of invariant edges of an EWG arises in the security analysis of statistical data, which will be discussed in section 6, and Gusfield [7] proved that if G is bipartite, then the set of invariant edges of Γ can be determined in time linear in the size of G. Here we present a linear time algorithm which finds the set of invariant edges of an arbitrary EWG.

2. Background. Let $G = (V, E)$ be a simple graph with vertex-edge incidence matrix $\mathbf{A}$. For any vector $\boldsymbol{x} = (x(e))_{e \in E}$, the *support* of $\boldsymbol{x}$ is the set $S = \{e \in E \colon x(e) \neq 0\}$, and the *signed support* of $\boldsymbol{x}$ is the ordered set pair (S^+, S^-), where $S^+ = \{e \in E \colon x(e) > 0\}$ and $S^- = \{e \in E \colon x(e) < 0\}$; moreover, the set $E - S$ is called the *zero set* of $\boldsymbol{x}$. The nonzero solutions of the homogeneous equation system $\mathbf{A}\boldsymbol{y} = \mathbf{0}$ are referred to as *circulations* in G and the linear space of the solutions of the homogeneous equation system $\mathbf{A}\boldsymbol{y} = \mathbf{0}$ is referred to as the *circulation space*.

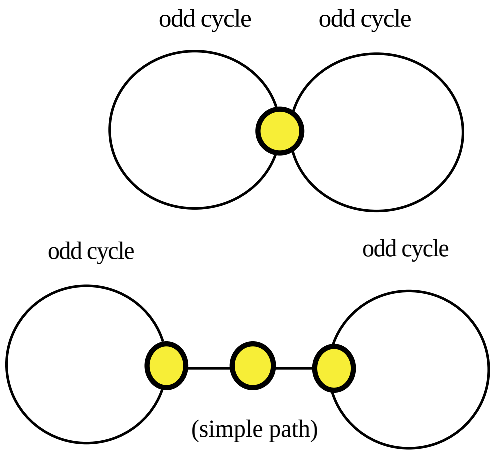

Fig. 3.

Thus, a nonempty subset S of E corresponds to a set of columns of $\mathbf{A}$ that are linearly dependent (over the field of reals) if and only if S contains the support of a circulation in G. A *minimal circulation* in G is a circulation in G with inclusion-minimal support. The following is a well-known result of linear algebra.

Proposition 1 (e.g., see page 107 in [3]). *Let S and (S^+, S^-) be the support and the signed support of a circulation in G, respectively. For each edge e in S, there is a minimal circulation in G with support C and signed support (C^+, C^-) such that e is in C, $C^+ \subseteq S^+$ and $C^- \subseteq S^-$.*

The set of supports of minimal circulations in G can be viewed as the family of *circuits* of a matroid [20], which we denote by $\boldsymbol{\mathcal{M}}(G)$, whose rank (i.e., the rank of $\mathbf{A}$) is given by $|V| - q$, where q is the number of connected components of G that are bipartite (in that they contain no odd cycles); see Theorem 1, page 421 in [6], or [19]. Explicitly, a subset of E is a circuit of $\boldsymbol{\mathcal{M}}(G)$ if and only if it is the edge set of either an even simple cycle or a pair of two edge-disjoint odd simple cycles that either have exactly one vertex in common or are vertex-disjoint and are connected by a simple path (see Figure 3) [5].

Let Z be a (proper or improper) subset of E. We say that a circuit of $\boldsymbol{\mathcal{M}}(G)$ is *Z-traversable* if it is the support of a (minimal) circulation whose signed support (C^+, C^-) is such that $Z \cap C^- = \emptyset$.

Consider now the vectors that are linear combinations of rows of $\mathbf{A}$. The inclusion-minimal supports of these vectors are the *cocircuits* of $\boldsymbol{\mathcal{M}}(G)$; that is, they are minimal edge sets whose removal decreases the rank of $\boldsymbol{\mathcal{M}}(G)$ [20]. Moreover, an edge e of G is a *coloop* of $\boldsymbol{\mathcal{M}}(G)$ if the singleton $\{e\}$ is a cocircuit of $\boldsymbol{\mathcal{M}}(G)$. In other words, an edge e of G is a coloop of $\boldsymbol{\mathcal{M}}(G)$ if and only if the incidence vector of $\{e\}$ is a linear combination of rows of $\mathbf{A}$ or, equivalently, if and only if e is not in any circuit of $\boldsymbol{\mathcal{M}}(G)$ [20].

3. Invariant edges.

In this section, we state a few characteristic properties of invariant edges of an arbitrary EWG which will be used later on. We need some preliminary definitions and results.

Let $\Gamma = (G, \boldsymbol{w})$ be an EWG with $G = (V, E)$ and let Z be the zero set of $\boldsymbol{w}$. A circulation in G with signed support (S^+, S^-) is said to be *legal* in Γ if $Z \cap S^- = \emptyset$. Accordingly, a circuit of $\boldsymbol{\mathcal{M}}(G)$ is Z-traversable if and only if it is the support of a (minimal) circulation in G which is legal in Γ. It should be noted that if the weights of the edges of G are all positive, then Z is empty so that each circulation in G is legal in Γ.

THEOREM 1. *Let $\Gamma = (G, \boldsymbol{w})$ be an EWG. An edge of G is not an invariant edge of Γ if and only if it belongs to the support of a circulation in G which is legal in Γ.*

Proof. ("only if") Let e be an edge of G that is not an invariant edge of Γ. Then, there exists a nonnegative solution $\boldsymbol{x}$ of (1) with $x(e) \neq w(e)$. The vector $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{w}$ is then a circulation in G. Let S and (S^+, S^-) be the support and the signed support of $\boldsymbol{y}$, respectively, and let Z be the zero set of $\boldsymbol{w}$. Then e is in S and, since $y(e') = x(e') \geq 0$ for each e' in Z, one has $Z \cap S^- = \emptyset$; that is, e belongs to the support of a circulation which is legal in Γ.

("if") Let $\boldsymbol{y}$ be a legal circulation with support S and signed support (S^+, S^-), and let e be in S. Consider the solution $\boldsymbol{x} = \boldsymbol{w} + \boldsymbol{y}$ of equation system (1). If $\boldsymbol{x}$ is nonnegative everywhere, then the statement follows from the fact that e is in S, which implies $x(e) \neq w(e)$. Otherwise, let

$$S' = \{e' \colon x(e') < 0\} \qquad \text{and} \qquad \lambda = \min\{-w(e')/y(e') \colon e' \in S'\}.$$

Since S' is a subset of S^- and $\boldsymbol{y}$ is a legal circulation, λ is positive. Then the vector $\boldsymbol{y}' = \lambda \boldsymbol{y}$ is a circulation in G having the same support and the same signed support as $\boldsymbol{y}$. Consider the solution $\boldsymbol{x}' = \boldsymbol{w} + \boldsymbol{y}'$ of equation system (1). It is easily seen that $\boldsymbol{x}'$ is nonnegative everywhere since, for each e' not in S, one has trivially $x'(e') \geq 0$ and, for each e' in S', one has

$$x'(e') = w(e') + \lambda y(e') = -y(e')(-w(e')/y(e') - \lambda) \geq 0.$$

Finally, since e is in the support S of $\boldsymbol{y}$, one has

$$x'(e) = w(e) + \lambda\, y(e) \neq w(e),$$

which proves the statement. □

THEOREM 2. *Let $\Gamma = (G, \boldsymbol{w})$ be an EWG and let Z be the zero set of $\boldsymbol{w}$. An edge of G is not an invariant edge of Γ if and only if it belongs to some Z-traversable circuit of $\boldsymbol{\mathcal{M}}(G)$.*

Proof. ("if") It follows from the "if" part of Theorem 1.

("only if") If e is not an invariant edge of Γ, then by the "only-if" part of Theorem 1 there is a circulation in G with support S and signed support (S^+, S^-) such that e is in S and $Z \cap S^- = \emptyset$. But, by Proposition 1, there is a minimal circulation in G such that its support contains e and its signed support (C^+, C^-) is such that $C^- \subseteq S^-$. Therefore, one has $Z \cap C^- \subseteq Z \cap S^- = \emptyset$, which proves the statement. □

EXAMPLE 1 (continued). *The zero set of $\boldsymbol{w}$ is $Z = \{(2,3), (2,5), (4,5)\}$. The minimal circulations in G are summarized in Figure 4 by taking λ to be any nonzero real. So, $\boldsymbol{\mathcal{M}}(G)$ contains one circuit which is not Z-traversable. By Theorem 2, each edge of G is an invariant edge of Γ.* □

EXAMPLE 2 (continued). *The zero set of $\boldsymbol{w}$ is $Z = \{(1,2), (1,3)\}$. The minimal circulations in G are summarized in Figure 5 by taking λ to be any nonzero real. So, $\boldsymbol{\mathcal{M}}(G)$ contains one circuit which is Z-traversable. By Theorem 2, no edge of G is an invariant edge of Γ.* □

Note that if the zero set Z of $\boldsymbol{w}$ is empty, then, by Theorem 2, an edge of G is an invariant edge of Γ if and only if it is not in any circuit of $\boldsymbol{\mathcal{M}}(G)$; that is, if and

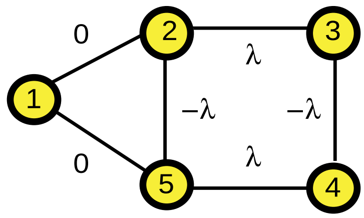

FIG. 4.

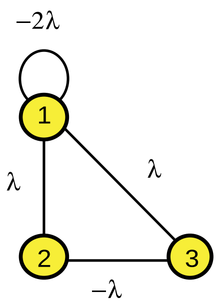

FIG. 5.

only if it is a coloop of $\boldsymbol{\mathcal{M}}(G)$. We now prove that the same holds in a more general case. In what follows, by the *kernel* of Γ [15] we mean the intersection of Z with the set of invariant edges of Γ □

LEMMA 1. *Let $\Gamma = (G, \boldsymbol{w})$ be an EWG whose kernel is empty. An edge of G is an invariant edge of Γ if and only if it is a coloop of $\boldsymbol{\mathcal{M}}(G)$.*

Proof. ("if") If an edge of G is a coloop of $\boldsymbol{\mathcal{M}}(G)$, then it is an invariant edge of Γ by the "only-if" part of Theorem 2.

("only if") Let e be an invariant edge of Γ. Suppose by contradiction that e is in some circuit of $\boldsymbol{\mathcal{M}}(G)$. Then, as is shown below, e should belong to the support of a legal circulation in Γ, which contradicts Theorem 1. To show that, suppose that e is in the circuit C_0 of $\boldsymbol{\mathcal{M}}(G)$. By the "only-if" part of Theorem 2, C_0 cannot be Z-traversable, where Z is the zero set of $\boldsymbol{w}$. Thus, if $\boldsymbol{c}_0$ is any minimal circulation in G with support C_0 and signed support (C_0^+, C_0^-), one has $Z \cap C_0^- \neq \emptyset$. Let $Z \cap C_0^- = \{e_1, \ldots, e_p\}$. Since the kernel of Γ is empty, no edge in Z is an invariant edge of Γ. Then, by the "only-if" part of Theorem 1, for each e_i, $1 \leq i \leq p$, there is a legal circulation $\boldsymbol{y}_i$ of Γ such that, if S_i is the support of $\boldsymbol{y}_i$, e_i is in S_i; moreover, if (S_i^+, S_i^-) is the signed support of $\boldsymbol{y}_i$, then $y_i(e_i) > 0$, since $Z \cap S_i^- = \emptyset$. Now consider the circulation

$$\boldsymbol{c}_i = [-c_0(e_i)/y_i(e_i)]\boldsymbol{y}_i.$$

Since $c_0(e_i) < 0$ (recall that $e_i \in C_0^-$) and $y_i(e_i) > 0$, $\boldsymbol{c}_i$ has the same support and signed support as $\boldsymbol{y}_i$, and hence is legal in Γ. Let

$$\boldsymbol{y} = \boldsymbol{c}_0 + \sum_{i=1,\ldots,p} \boldsymbol{c}_i.$$

Since the circulation space of $\mathbf{A}$ is a linear space, $\boldsymbol{y}$ is still a circulation in G. Let S and (S^+, S^-) be the support and signed support of $\boldsymbol{y}$, respectively. Finally, we now prove that (i) e is in S, and (ii) the circulation $\boldsymbol{y}$ is legal in Γ.

Proof of (i). Since e is an invariant edge of Γ, by the "if" part of Theorem 2, e is in the support of none of the legal circulations $\boldsymbol{c}_i$ so that $c_i(e) = 0$ for each i, $1 \leq i \leq p$. Therefore, $y(e) = c_0(e)$ and, since e is in C_0, one has that e is also in S.

Proof of (ii). In order to prove that the intersection of Z with S^- is empty, we separately examine the edges $e_1, \dots, e_p$ in $Z \cap C_0^-$ and the edges in $Z - C_0^-$. For each i and j, $1 \leq i, j \leq p$, $c_j(e_i) \geq 0$, since $Z \cap S_j^- = \emptyset$. Moreover, for each i, $1 \leq i \leq p$, one has

$$c_i(e_i) = -c_0(e_i)$$

and, hence,

$$y(e_i) = c_0(e_i) + c_i(e_i) + \sum_{j \neq i} c_j(e_i) = \sum_{j \neq i} c_j(e_i) \geq 0.$$

Therefore, each e_i is not in S^-.

We now consider the edges in $Z - C_0^-$. If e' is such an edge, then $c_0(e') \geq 0$. Moreover, since $Z \cap S_i^- = \emptyset$, $1 \leq i \leq p$, one has $c_i(e') \geq 0$. Therefore

$$y(e') = c_0(e') + \sum_{i=1,\dots,p} c_i(e') \geq 0,$$

and, hence, e' is not in S^-.

After proving (i) and (ii), by the "if" part of Theorem 1, one has that e is not an invariant edge of Γ (a contradiction). $\square$

As a consequence of Lemma 1, we obtain the following characterization of invariant edges of an EWG.

THEOREM 3. *Let $\Gamma = (G, \boldsymbol{w})$ be an EWG with kernel K. The set of invariant edges of Γ is the union of K with the set of coloops of $\boldsymbol{\mathcal{M}}(G - K)$.*

Proof. Let $\Gamma' = (G', \boldsymbol{w}')$, where $G' = G - K$, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to the edge set of G'. It is clear that an edge of G is an invariant edge of Γ if and only if either it is in K or it is an invariant edge of Γ'. On the other hand, the kernel of Γ' is empty so that, by Lemma 1, the invariant edges of Γ' are exactly the coloops of $\boldsymbol{\mathcal{M}}(G')$. $\square$

By Theorem 3, the set of invariant edges of $\Gamma = (G, \boldsymbol{w})$ can be found by determining first the kernel K of Γ and, then, the set of coloops of $\boldsymbol{\mathcal{M}}(G - K)$. We shall solve the problem of the kernel of an EWG in section 5 and, in the next section, we shall give a linear algorithm for finding the set of coloops of the matroid on a graph.

4. Finding the coloop set.

Let $G = (V, E)$ be a simple graph. Bearing in mind that a subset of E is a cocircuit of $\boldsymbol{\mathcal{M}}(G)$ if and only if it is a minimal edge set whose removal decreases the rank of $\boldsymbol{\mathcal{M}}(G)$, one easily obtains the following proposition.

PROPOSITION 2 (see [5]). *An edge of G is a coloop of $\boldsymbol{\mathcal{M}}(G)$ if and only if its removal creates one more bipartite connected component.*

Let e be a coloop of $\boldsymbol{\mathcal{M}}(G)$. The graph $G - e$ has or has not one more connected component than G. By Proposition 2, in the former case e must be a bridge, which we call an *algebraic bridge* of G, and in the latter case, as is shown below, e is an *odd edge*, by which we mean that e is common to all odd cycles of the connected component G containing e.

LEMMA 2. *An edge of a simple graph G is a coloop of $\boldsymbol{\mathcal{M}}(G)$ if and only if it is either an algebraic bridge or an odd edge.*

Proof. The statement is trivial if the graph is bipartite since, by Proposition 2, each coloop of $\mathcal{M}(G)$ is a bridge and vice versa. Consider now a graph G which is not bipartite. Without loss of generality, we assume G is connected. It is sufficient to prove that a coloop e of $\mathcal{M}(G)$ is not a bridge if and only if it is an odd edge. If e is not a bridge, then, by Proposition 2, $G - e$ is bipartite and connected and, hence, every odd cycle of G must contain e; that is, e is an odd edge of G. On the other hand, if e is an odd edge of G, then $G - e$ is connected and contains no odd cycles so that, by Proposition 2, e is a coloop of $\mathcal{M}(G)$. ∎

EXAMPLE 1 (continued). *The coloops of $\mathcal{M}(G)$ are the two edges missing from the even simple cycle supporting the minimal circulations shown in Figure* 4*. Both of them are odd edges.* ∎

EXAMPLE 2 (continued). *$\mathcal{M}(G)$ has no coloops (see Figure* 5*).* ∎

Let $G = (V, E)$ be a simple graph which without loss of generality we assume to be connected. We first show that the problem of finding the set of coloops of $\mathcal{M}(G)$ is polynomial; next, we shall give a linear algorithm based on Lemma 2.

In [14, 16, 17] an $O(|E|)$ algorithm is given to decide whether the incidence vector of a given subset of E is orthogonal to the space of circulations in G. By applying that algorithm to each singleton, one can determine the set of coloops of $\mathcal{M}(G)$ in $O(|E|^2)$ time. In the next two subsections, we give two linear algorithms for finding the algebraic bridges and the odd edges of G; so, by Lemma 2, determining the whole set of coloops of $\mathcal{M}(G)$ requires $O(|E|)$ time.

4.1. Algebraic bridges. Let $G = (V, E)$ be a connected simple graph, and let B be the set of bridges of G. Consider the tree $T = (N, A)$ whose nodes represent the connected components of $G - B$ and whose arcs represent the bridges of G. A node n of T is marked if the corresponding connected component of $G - B$ is not bipartite. If no node of T is marked, then G is bipartite and the bridges of G are all and the only algebraic bridges. Otherwise, there is at least one marked node of T; then, arbitrarily choose a marked node r of T and let T_r be the directed tree obtained by rooting T at r. For each node n of T_r, $n \neq r$, let $par(n)$ be the parent of n in T_r. Of course, a bridge of G is algebraic if and only if the (directed) arc $\langle par(n), n\rangle$ of T_r is such that the subtree of T_r rooted at n contains no marked nodes. Thus, in order to get the algebraic bridges of G, it is sufficient to perform a postorder traversal of T_r[1]: when node n is examined, $n \neq r$; if n is marked, then the edge of G corresponding to the arc $\langle par(n), n\rangle$ is removed from B and the vertex $par(n)$ is marked if it was unmarked. So, the ultimate value of B is exactly the set of algebraic edges of G. Now, since the construction of T and B and the postorder traversal of T_r require $O(|E|)$ time, we have the following theorem.

THEOREM 4. *The set of algebraic bridges of a connected simple graph can be found in time linear in the number of its edges.*

4.2. Odd edges. Let $G = (V, E)$ be a connected simple graph. Trivially, if G is bipartite, then G contains no odd edges. In the case where G is not bipartite, we shall show that the set of odd edges of G can be found in $O(|E|)$. To achieve this, we need the following technical lemmas, the first two of which refer to general properties of the symmetric difference ($\oplus$) of cycles.

LEMMA 3 (see, e.g., [1]). *The symmetric difference of two distinct nondisjoint cycles is a set of edge-disjoint cycles.*

LEMMA 4. *If the symmetric difference of two or more cycles contains an odd number of edges, then the number of such cycles having odd lengths is odd.*

Proof. It easily follows from the fact that, for every two sets C and C', $|C \oplus C'|$ is odd if and only if $|C|$ and $|C'|$ have different parities. □

Let T be the edge set of a spanning tree of G. For each back-edge e (i.e., e not in T), the set $T \cup [e]$ contains exactly one simple cycle; such simple cycles, one for each back-edge, are called the *fundamental cycles* of G with respect to T [18].

LEMMA 5 (see, e.g., page 251 in [1]). *Let T be the edge set of a spanning tree of a simple graph G. Every cycle of G can be expressed as a symmetric difference of fundamental cycles of G with respect to T.*

LEMMA 6. *Let G be a nonbipartite connected simple graph and let T be a spanning tree of G. An edge of G is an odd edge of G if and only if it is in all odd fundamental cycles with respect to T and in no even fundamental cycle with respect to T.*

Proof. ("if") Let e be an edge of G that is in all odd fundamental cycles with respect to T and in no even fundamental cycle with respect to T. Let C be any odd cycle. By Lemma 5, C can be expressed as symmetric difference of fundamental cycles with respect to T, and, by Lemma 4, the number of odd fundamental cycles in its expression is odd so that, since e is in all of them and in no even fundamental cycle with respect to T, e belongs to C.

("only if") Let e be an odd edge of G. Of course e is in all odd fundamental cycles with respect to T. Suppose by contradiction that there is an even fundamental cycle C' with respect to T that contains e. Let C be an odd cycle containing e. By Lemma 3, $C \oplus C'$ contains an odd cycle, say C'', because the lengths of C and C' have different parities. So, since e is in both C and C', e is not in C'', which contradicts the hypothesis that e is in all odd cycles of G. □

From a computational point of view, the fundamental cycles of G with respect to a given spanning tree can be constructed using an $O(|V|^3)$ algorithm (see, e.g., Algorithm 8.10 in [18]). So, by Lemma 6 one can resort to that algorithm to find the set of odd edges of G in $O(|V|^3)$ time. However, we shall use Lemma 6 to work out an algorithm which runs in $O(|E|)$ time. It consists of two phases.

Phase I. Arbitrarily choose a vertex r of G and perform a traversal of G with the depth-first search (DFS) technique to produce

— the edge set T of a directed spanning tree of G,

— the set B of back-edges that create odd fundamental cycles of G with respect to T, and

— a vertex table which, for each vertex v, reports the following information items:
  - the DFS number of v, denoted by $n(v)$;
  - a label, denoted by $col(v)$, which is set to "white" or "black" depending on whether the length of the path from r to v in the spanning tree is even or odd;
  - if $v \neq r$, the parent of v, denoted by $par(v)$;
  - if $v \neq r$, the tree-edge $\langle par(v), v\rangle$, denoted by $arc(v)$.

Phase II. First of all, join a back-edge to *Odd* if it is the unique element of B. Next, in order to decide if a tree-edge e can be joined to *Odd*, compute

— the number of the even fundamental cycles that contain e, denoted by NEC[e], and

— the number of the odd fundamental cycles that contain e, denoted by NOC[e],

as follows. For each vertex u, let $N(u)$ be the set of neighbors of u in G and let $C(u)$ be the set of children of u in T. Then, set (see Figure 6)

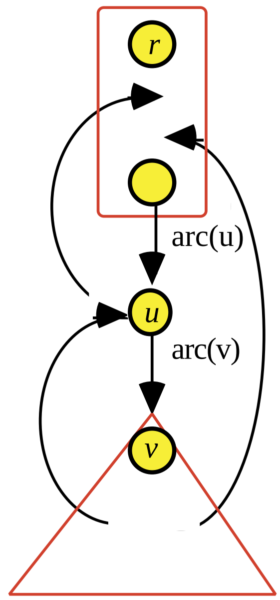

Fig. 6.

$$\text{NEC}[arc(u)] = |P_{even}(u)| + \sum_{v \in C(u)} \text{NEC}[arc(v)] - |S_{even}(u)|, \tag{2}$$

where

$$P_{even}(u) = \{v \in N(u)\colon par(u) \neq v \text{ and } col(v) \neq col(u) \text{ and } n(v) < n(u)\},$$
$$S_{even}(u) = \{v \in N(u)\colon par(v) \neq u \text{ and } col(v) \neq col(u) \text{ and } n(v) > n(u)\},$$

and

$$\text{NOC}[arc(u)] := |P_{odd}(u)| + \sum_{v \in C(u)} \text{NOC}[arc(v)] - |S_{odd}(u)|, \tag{3}$$

where

$$P_{odd}(u) = \{v \in N(u)\colon par(u) \neq v \text{ and } col(v) = col(u) \text{ and } n(v) < n(u)\},$$
$$S_{odd}(u) = \{v \in N(u)\colon par(v) \neq u \text{ and } col(v) = col(u) \text{ and } n(v) > n(u)\}.$$

After calculating the quantities NEC[e] and NOC[e] for each edge e in T, determine the set of odd edges, denoted by *Odd*, as follows (see Lemma 6): for each edge e in T, join e to *Odd* if NEC[e] $= 0$ and NOC[e] $= |B|$.

The following algorithm details the steps of Phase II.

Algorithm 1.

Input: A nonbipartite, connected simple graph $G = (V, E)$, a vertex r of G, T, B and the vertex table of G.

Output: The set *Odd* of odd edges of G.

(1) Set $Odd := \emptyset$. Set $k := |B|$. If $k = 1$, then $Odd := Odd \cup B$.
For each edge e in T, set NEC[e] := NOC[e] := 0.

(2) For each child u of r in T, TRAVERSE (G, u).

(3) For each edge e in T, if NEC$[e] = 0$ and NOC$[e] = k$, then add e fo *Odd*.
PROCEDURE TRAVERSE (G, u).
For each neighbor v of u, do:
begin
If v is a child of u, then do:
begin
TRAVERSE (G, v);
NEC$[arc(u)]$:= NEC$[arc(u)]$ + NEC$[arc(v)]$;
NOC$[arc(u)]$:= NOC$[arc(u)]$ + NOC$[arc(v)]$
end;
otherwise, if v is not the parent of u, then do:
Case 1. if $n(v) > n(u)$ and $col(v) \neq col(u)$, then set

$$\mathrm{NEC}[arc(u)] := \mathrm{NEC}[arc(u)] - 1;$$

Case 2. if $n(v) > n(u)$ and $col(v) = col(u)$, then set

$$\mathrm{NOC}[arc(u)] := \mathrm{NOC}[arc(u)] - 1;$$

Case 3. if $n(v) < n(u)$ and $col(v) \neq col(u)$, then set

$$\mathrm{NEC}[arc(u)] := \mathrm{NEC}[arc(u)] + 1;$$

Case 4. if $n(v) < n(u)$ and $col(v) = col(u)$, then set

$$\mathrm{NOC}[arc(u)] := \mathrm{NOC}[arc(u)] + 1$$

end.

THEOREM 5. *Let G be a nonbipartite, connected simple graph. The value of Odd computed by Algorithm 1 with input G and vertex r is exactly the set of odd edges of G.*

Proof. It is sufficient to prove that the quantities NEC$[e]$ and NOC$[e]$, for each tree-edge e, equal the number of even fundamental cycles containing e and the number of odd fundamental cycles containing e, respectively. The statement is proven by structural induction.

Basis. Assume that u is a leaf of T. Then, u has no children so that if v is a neighbor of u, then v must be an ancestor of u. If $v = u$, then the self-loop (u, u) contributes to neither NEC$[arc(u)]$ nor NOC$[arc(u)]$. If v is a proper ancestor of the parent of u, then $n(v) < n(u)$ and the back-edge (u, v) correctly adds 1 to either NEC$[arc(u)]$ or NOC$[arc(u)]$.

Inductive step. Let u be not a leaf of T and assume the statement holds for each one of the children of u. Thus, if v is a child of u, then values of both NEC$[arc(v)]$ and NOC$[arc(v)]$ are right. It is then easily seen that, by formulae (2) and (3), the statement also holds for u. $\square$

From the complexity-theoretic point of view, it is easily seen that the time of Algorithm 1 is dominated by the time required by the DFS traversal and, hence, is $O(|E|)$. So, by Theorem 5 one has the following corollary.

COROLLARY 1. *Let $G = (V, E)$ be a nonbipartite, connected simple graph. The set of odd edges of G can be found in $O(|E|)$ time.*

5. Finding the kernel. Let $\Gamma = (G, \boldsymbol{w})$ be an EWG with $G = (V, E)$ and kernel K. If the zero set Z of $\boldsymbol{w}$ is empty, then K is empty too and we are done. Assume that

Z is not empty. If G is bipartite, Gusfield [7] proved that K equals the set of directed edges joining strongly connected components of the *mixed graph* $G(Z)$ obtained from G by directing all the edges in Z from one side of the bipartition to the other one so that it can be computed in time linear in the size of G. In this section we show that, even in the case where G is not bipartite, the kernel of Γ can be computed in time linear in the size of G.

With Γ we associate a bipartite EWG $\Gamma' = (G', \boldsymbol{w}')$, which we call a *bipartite EWG associated* with Γ. The graph $G' = (V', E')$ is constructed as follows. Let B be a maximal bipartite partial graph of G and let $\{V_1, V_2\}$ be a bipartition of V such that each edge of B has one end in V_1 and the other end in V_2. Let $\overline{V}$ be a "copy" of V, that is, $\overline{V} \cap V = \emptyset$ and $|\overline{V}| = |V|$. If v is a vertex of G, then by $\bar{v}$ we denote the copy of v. The vertex set of G' is taken to be $V' = V \cup \overline{V}$, and the edge set of G' is taken to be

$$E' = \bigcup_{e \in E} f(e),$$

where f is function defined on E as follows:
— if e is a self-loop, say (v, v), then $f(e) = \{(v, \bar{v})\}$;
— if $e = (u, v)$ is an edge of B, then $f(e) = \{(u, v), (\bar{u}, \bar{v})\}$;
— if $e = (u, v)$ is neither a self-loop nor an edge of B, then $f(e) = \{(u, \bar{v}), (\bar{u}, v)\}$.

The set $f(e)$ will be referred to as the *image* of e in G'. Let $V_1' = V_1 \cup \overline{V}_2$ and $V_2' = \overline{V}_1 \cup V_2$, where $\overline{V}_1 = \{\bar{v} : v \in V_i\}$, $i = 1, 2$. The graph G' is bipartite and the partition $\{V_1', V_2'\}$ of V' is such that each edge of G' has one end in V_1' and the other end in V_2'. Furthermore, G' is connected if and only if G is not bipartite. Finally, to each edge e' of G' we assign the weight $w'(e') = w(e)$, where e is the edge of G for which $e' \in f(e)$. Let $\mathbf{A}'$ be the incidence matrix of G' and let $\boldsymbol{b}' = (b'(v'))_{v' \in V'}$, where $b'(v')$ equals the sum of the weights $w'(e')$ of the edges of G' incident to v'. Consider the equation system

$$\mathbf{A}'\boldsymbol{x}' = \boldsymbol{b}'. \tag{4}$$

For every edge e' of G', let $L[x'(e')]$ and $U[x'(e')]$ denote the minimum and the maximum of the variable $x'(e')$ over the nonnegative solutions of equation system (4), respectively. Moreover, for every edge e of G, let $L[f(e)]$ and $U[f(e)]$ denote the minimum and the maximum of the expression $\sum_{e' \in f(e)} x'(e')$ over the nonnegative solutions of equation system (4), respectively. First, observe that if $\boldsymbol{x}'$ is a (nonnegative) solution of equation system (4), then a (nonnegative) solution $\boldsymbol{x}''$ of equation system (4) can be obtained by setting for each edge e' of G'

$$x''(e') = x'(e') \text{ if } \{e'\} \text{ is the image of a self-loop of } G$$

and

$$x''(e') = x'(e'') \text{ if } \{e', e''\} \text{ is the image of an edge of } G \text{ that is not a self-loop.}$$

It follows that, if $\{e', e''\}$ is the image of an edge of G that is not a self-loop, then

$$L[x'(e')] = L[x'(e'')] \quad \text{and} \quad U[x'(e')] = U[x'(e'')]. \tag{5}$$

Second, if $\boldsymbol{x}$ is a (nonnegative) solution of equation system (1), then a (nonnegative) solution $\boldsymbol{x}'$ of equation system (4) can be obtained by setting for each edge e' of G'

$$x'(e') = x(e), \text{ where } e \text{ is the edge of } G \text{ whose image } f(e) \text{ contains } e'.$$

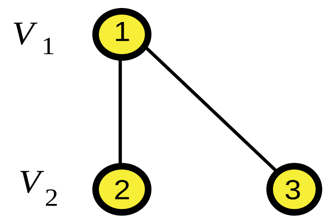

FIG. 7.

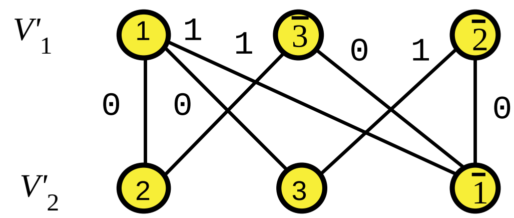

FIG. 8.

On the other hand, if $\boldsymbol{x}'$ is a (nonnegative) solution of equation system (4), then a (nonnegative) solution $\boldsymbol{x}$ of equation system (1) can be obtained by setting for each edge e of G

$$x(e) = \left[\sum_{e' \in f(e)} x'(e') \right] \Big/ |f(e)|.$$

Therefore, one has

$$L[x(e)] = (1/|f(e)|)L[f(e)] \quad \text{and} \quad U[x(e)] = (1/|f(e)|)U[f(e)]. \tag{6}$$

EXAMPLE 2 (continued). *By choosing as the maximal bipartite partial graph of G the graph shown in Figure 7, we associate with Γ the bipartite EWG $\Gamma' = (G', \boldsymbol{w}')$ shown in Figure 8.*

The general expression of a nonnegative solution of equation system (4) is

$$\begin{aligned}
x'(1,2) &= x'(\bar{1},\bar{3}) = \mu, \\
x'(1,3) &= x'(\bar{1},\bar{2}) = \nu, \\
x'(1,\bar{1}) &= 1 - \mu - \nu, \\
x'(2,\bar{3}) &= 1 - \mu, \\
x'(\bar{2},3) &= 1 - \nu,
\end{aligned}$$

where μ and ν are bounded as shown in Figure 9. At this point, it is easy to check formulae (5) and (6). $\square$

We now state some technical results to relate the kernels of Γ and Γ'.

LEMMA 7. *Let $\Gamma = (G, \boldsymbol{w})$ be an EWG and let $\Gamma' = (G', \boldsymbol{w}')$ be a bipartite EWG associated with Γ. An edge e of G is an invariant edge of Γ if and only if $L[f(e)] = U[f(e)]$, where $f(e)$ is the image of e in G'.*

Proof. The proof follows from formula (6). $\square$

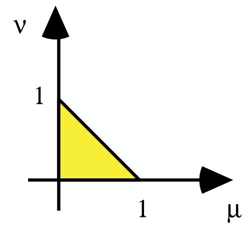

FIG. 9.

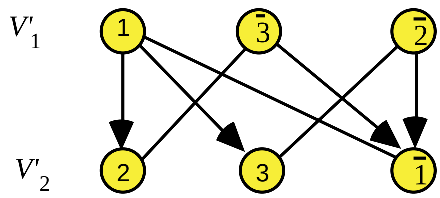

FIG. 10.

LEMMA 8. *Let $\Gamma = (G, \boldsymbol{w})$ be an EWG and let $\Gamma' = (G', \boldsymbol{w}')$ be a bipartite EWG associated with Γ. An edge e of G belongs to the kernel of Γ if and only if its image in G' is contained in the kernel of Γ'.*

Proof. If e is a self-loop of G, then the statement immediately follows from formula (6) and Lemma 7. We now prove the statement in the case where e is not a self-loop and $f(e) = \{e', e''\}$.

("if") If both e' and e'' belong to the kernel of Γ', then $x'(e') = x'(e'') = 0$ for every solution $\boldsymbol{x}'$ of equation system (4). Therefore, $L[x'(e') + x'(e'')] = U[x'(e') + x'(e'')] = 0$ and the statement follows from formula (6).

("only if") If e belongs to the kernel of Γ, then, by formula (6), one has

$$x'(e') + x'(e'') = 0$$

for every nonnegative solution $\boldsymbol{x}'$ of equation system (4). By the nonnegativity of $\boldsymbol{x}'$, $x'(e') = x'(e'') = 0$, which proves that both e' and e'' belong to the kernel of Γ'. ∎

COROLLARY 2. *Let $\Gamma = (G, \boldsymbol{w})$ be an EWG and let $\Gamma' = (G', \boldsymbol{w}')$ be a bipartite EWG associated with Γ'. An edge of G belongs to the kernel of Γ if and only if an element of its image in G' belongs to the kernel of Γ'.*

Proof. The proof follows from Lemma 8 and formula (5). ∎

EXAMPLE 2 (continued). *The zero set of $\boldsymbol{w}'$ is $Z' = \{(1,2), (1,3), (\bar{1},\bar{2}), (\bar{1},\bar{3})\}$. The mixed graph $G'(Z')$ is strongly connected (see Figure 10). So, the kernel of Γ' is empty. By Corollary 2, the kernel of Γ is empty.* ∎

THEOREM 6. *The kernel of an EWG can be found in time linear in the size of G.*

Proof. Let $\Gamma = (G, \boldsymbol{w})$ be an EWG and let $\Gamma' = (G', \boldsymbol{w}')$ be a bipartite EWG associated with Γ. If G is bipartite, then the statement was proven by Gusfield [7]. Otherwise, since G' is bipartite, the kernel K' of Γ' can be found in time linear in the size of G' and, hence, of G. So, it is sufficient to prove that both constructing G' and determining K from K' take a linear time. In order to construct G', we perform a DFS traversal of G, which allows us to find both a maximal bipartite partial graph B of G and the nontree edges that create odd cycles when added to B. When an edge

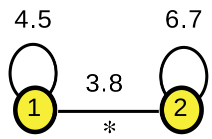

FIG. 11.

e' of G' is created, we get e' to point to the edge e of G for which $e' \in f(e)$. Finally, by Corollary 2, the set K can be obtained as follows. Initially, each edge e of G is unmarked. For each element e' of K', if the edge e of G that e' points to is unmarked, then e is marked and added to K. ∎

6. Security of statistical data. In the security analysis of statistical data [4, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17], EWGs and, more in general, weighted hypergraphs can be used to control the amount of information that is implicitly released when statistical data are made public, in order to avoid disclosure of confidential data. We now illustrate this application by discussing a typical case. Suppose that we are given a data set $\{b_i : i \in I\}$, where b_i is the value of a confidential variable b of nonnegative real type (e.g., salary) for individual i in the population I. The sum of values of b over a subset I' of I is called *sensitive* if I' contains exactly one individual. Now, given a statistical summary $\sigma = \{b(v) : v \in V\}$ where $b(v)$ is the sum of b over a subset $I(v)$ of I containing at least two people, for each v in V, the problem that naturally arises consists of checking that no sensitive sum is implicitly released. We now present a graph-theoretic approach to this problem. Let $I_\sigma = \cup_{v \in V} I(v)$, which we call the set of individuals *covered* by σ. The *basic partition* of I_σ is the coarsest partition of I_σ such that each $I(v)$ can be obtained as the union of one or more classes of the partition. A class J of the basic partition of I_σ will be indexed by the set $e = \{v \in V : J \subseteq I(v)\}$. If E is the index set of the classes of the basic partition of I_σ, the pair $G = (V, E)$ defines a hypergraph where hyperedge e is incident to vertex v if and only if v belongs to e. Consider the weighted hypergraph $\Gamma = (G, \boldsymbol{w})$, where, for each hyperedge e of G, $w(e)$ is given by the sum of the values of the variable b over the class $J(e)$ of the basic partition of I_σ indexed by e; that is,

$$w(e) = \sum_{i \in J(e)} b_i.$$

Finally, a hyperedge e of G is *marked* if $|J(e)| = 1$. Then, no sensitive sum is implicitly released given σ if and only if no invariant hyperedge of Γ is marked. If this is the case, the statistical summary σ is said to be *safe*. Since the invariant edges of an EWG can be found in linear time, one has that, if G is a graph, then one can decide whether σ is or is not safe in linear time, too.

EXAMPLE 3. *Consider five individuals with salaries $b_1 = 2.0$, $b_2 = 2.5$, $b_3 = 3.8$, $b_4 = 3.7$, and $b_5 = 3.0$. Suppose that the two sums $b_1 + b_2 + b_3$ and $b_3 + b_4 + b_5$ are made public. Let $\sigma_1 = \{b_1 + b_2 + b_3, b_3 + b_4 + b_5\}$. The set of individuals covered by σ_1 is $\{1, \ldots, 5\}$ and its basic partition consists of the three classes $\{1, 2\}$, $\{3\}$, and $\{4, 5\}$. Thus, the weighted hypergraph Γ_1 associated with σ_1 is the EWG shown in Figure 11, where the edge $(1, 2)$ is marked. Since the set of invariant edges of Γ_1 turns out to be empty, σ_1 is safe. Next, suppose that the sum $b_1 + b_2 + b_4 + b_5$ is also made public.*

Let $\sigma_2 = \{b_1 + b_2 + b_3, b_3 + b_4 + b_5, b_1 + b_2 + b_4 + b_5\}$. Again, the set of individuals covered by σ_2 is $\{1, \ldots, 5\}$ and its basic partition consists of the three classes $\{1, 2\}$,

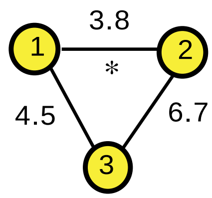

Fig. 12.

$\{3\}$, *and* $\{4,5\}$. *The weighted hypergraph* Γ_2 *associated with* σ_2 *is the EWG shown in Figure* 12, *where the edge* $(1,2)$ *is marked. Since each edge is an invariant edge of* Γ_2, σ_2 *is not safe (and the salary* b_3 *is unprotected).* ∎

Let $\sigma = \{b(v)\colon v \in V\}$ be a statistical summary of the data set $\{b_i\colon i \in I\}$ and let $\Gamma = (G, \boldsymbol{w})$ be the associated weighted hypergraph. It is worth noting that if v is a "leaf" of G, that is, if v belongs to exactly one hyperedge e of G, then the class of the basic partition of I_σ indexed by e coincides with $I(v)$ so that $w(e) = b(v)$; furthermore, the hyperedge e is definitely an invariant hyperedge of Γ, and, since $|I(v)| > 1$, it is not marked. As we are interested in checking the existence of marked invariant hyperedges of Γ (if any), we can reduce Γ by deleting all leaves of G and their incident hyperedges. Let $\Gamma' = (G', \boldsymbol{w}')$ be the resulting weighted hypergraph. Of course, σ is safe if and only if no invariant hyperedge of Γ' is sensitive. We now show that if σ is a two-dimensional table with suppressions, then G' is always a graph so that one can decide whether σ is or is not safe in linear time. Let σ be obtained from a complete two-dimensional table T by suppressing all sensitive cells ("primary suppressions") as well as additional (internal or marginal) cells to exclude the possibility of arriving at the contents of sensitive cells by indirect methods ("complementary suppressions"). Denote by

— $T(r,c)$ the value of internal cell (r,c), $1 \le r \le m$, and $1 \le c \le n$,
— $T(r,+)$ the rth row total, $1 \le r \le m$, and
— $T(+,c)$ the cth column total, $1 \le c \le n$.

Assume that each $T(r,c)$ is the sum of the values of a confidential variable of non-negative real type over the set $I(r,c)$ of individuals. So, a cell (r,c) of T is sensitive if $|I(r,c)| = 1$. We first detail the structure of the weighted hypergraph $\Gamma = (G, \boldsymbol{w})$ associated with σ and then show that the reduction of Γ results in an EWG. Let U, R, and C be the set of unsuppressed internal cells, the set of marginal cells corresponding to unsuppressed row totals, and the set of marginal cells corresponding to unsuppressed column totals, respectively. Then the vertex set of G is

$$V = U \cup R \cup C.$$

Let $S = \{(r,c) \notin U\colon\ r \notin R \text{ and } c \notin C\}$. Moreover, for each $r \in R$, let $C_r = \{c \notin C\colon (r,c) \notin U\}$; analogously, for each $c \in C$, let $R_c = \{r \notin R\colon\ (r,c) \notin U\}$. Then, the set of individuals covered by σ is $I_\sigma = \cup_{(r,c)\notin S} I(r,c)$ and the basic partition of I_σ contains

— one class $I(r,c)$ for each (r,c) in U and for each (r,c) not in U with $r \in R$ and $c \in C$,
— one class $\cup_{c \in C_r} I(r,c)$ for each $r \in R$ with $C_r \ne \emptyset$, and
— one class $\cup_{r \in R_c} I(r,c)$ for each $c \in C$ with $R_c \ne \emptyset$.

Recall that the hyperedges of G are the indices of these classes. The hyperedge e

indexing a class such as $I(r, c)$ is

$$\begin{array}{ll} e = \{(r,c),(r,+),(+,c)\} & \text{if } (r,c) \in U, r \in R, c \in C, \\ e = \{(r,c),(r,+)\} & \text{if } (r,c) \in U, r \in R, c \notin C, \\ e = \{(r,c),(+,c)\} & \text{if } (r,c) \in U, r \notin R, c \in C, \\ e = \{(r,c)\} & \text{if } (r,c) \in U, r \notin R, c \notin C, \\ e = \{(r,+),(+,c)\} & \text{if } (r,c) \notin U, r \in R, c \in C, \end{array}$$

and $w(e)$ is always set to $T(r, c)$. For the hyperedge e indexing a class such as $\cup_{c \in C_r} I(r, c)$, one has $e = \{(r, +)\}$ and

$$w(e) = \sum_{c \in C_r} T(r, c),$$

and for the hyperedge e indexing a class such as $\cup_{r \in R_c} I(r, c)$, one has $e = \{(+, c)\}$ and

$$w(e) = \sum_{r \in R_c} T(r, c).$$

At this point, it should be clear that the leaves of G are all and the only vertices of the type (r, c), of the type $(r, +)$ with $r \in R$ and $C_r = \{1, \ldots, n\}$, and of the type $(+, c)$ with $c \in C$ and $R_c = \{1, \ldots, m\}$. Let L be the set of leaves of G and let $R' = R - L$ and $C' = C - L$. After deleting all the leaves of the hypergraph G, we remain with the hypergraph $G' = (V', E')$ whose hyperedges are incident to at most two vertices. More precisely, one has that $V' = R' \cup C'$ and E' consists of the edges

$$\begin{array}{ll} \{(r,+),(+,c)\} & \text{if } (r,c) \notin U, r \in R, c \in C, \\ \{(r,+)\} & \text{with } r \in R', \\ \{(+,c)\} & \text{with } c \in C'. \end{array}$$

To sum up, the reduction of the weighted hypergraph associated with σ is an EWG and, therefore, the safety of σ can be tested in linear time.

EXAMPLE 4. *Consider the table of Figure* 13 *whose entries are assumed to be nonnegative reals. Suppose that the following cells*

$$(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (4, 4)$$

are all sensitive. The table of Figure 14 *is obtained from the table of Figure* 13 *by suppressing all the six sensitive cells and the additional cells* $(3, 3)$*,* $(3, +)$*,* $(4, +)$*, and* $(+, 4)$*.*

The reduced weighted hypergraph $\Gamma' = (G', \boldsymbol{w}')$ *associated with the table of Figure* 14 *is the EWG shown in Figure* 15*. The invariant edges of* Γ' *are the edge joining the vertices* $(2, +)$ *and* $(+, 3)$ *and the self-loop at vertex* $(+, 3)$*. One of these two edges is marked and, therefore, the table of Figure* 14 *is not safe.* □

7. Closing remarks. We solved the problem of finding the set of invariant edges of an EWG under the assumption that edge weights are nonnegative reals. The case where edge weights are nonnegative integers is an open problem. However, if the underlying graph of the EWG is bipartite, then Gusfield's algorithm still holds owing to the total unimodularity of the incidence matrix.

	j=1	j=2	j=3	j=4	
i=1	0	10	0	20	T(1,+) = 30
i=2	2	3	0	20	T(2,+) = 25
i=3	17	13	5	0	T(3,+) = 35
i=4	16	15	14	5	T(4,+) = 50
	T(+,1) = 35	T(+,2) = 41	T(+,3) = 19	T(+,4) = 45	

Fig. 13.

	j=1	j=2	j=3	j=4	
i=1			0	20	T(1,+) = 30
i=2				20	T(2,+) = 25
i=3	17	13		0	
i=4	16	15	14		
	T(+,1) = 35	T(+,2) = 41	T(+,3) = 19		

Fig. 14.

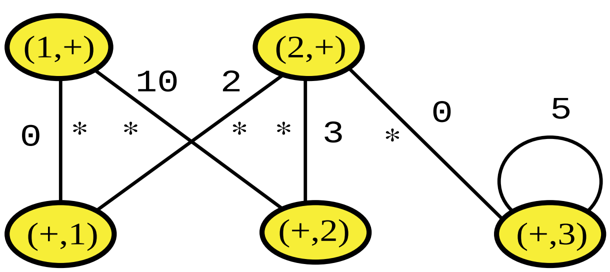

Fig. 15.

A natural generalization of the problem dealt with in this paper is the search of invariant edges of an edge-weighted hypergraph. It should be noted that mutatis mutandis Theorem 3 (see section 3) applies to edge-weighted hypergraphs, too. So, in order to find the invariant edges of an edge-weighted hypergraph $(G, \boldsymbol{w})$, we have to devise a procedure for computing its kernel, say K, and the coloops of the matroid $\boldsymbol{\mathcal{M}}(G-K)$. It should be clear that, in order to find to coloops of $\boldsymbol{\mathcal{M}}(G-K)$, we need a formula for the rank of the incidence matrix of G. At present, such a formula is known only for special classes of hypergraphs, e.g., for the class of connected uniform hypergraphs [2].

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