The Extremal Function for 3-linked Graphs

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Abstract

A graph is *k-linked* if for every set of $2k$ distinct vertices $\{s_1, \dots, s_k, t_1, \dots, t_k\}$ there exist disjoint paths $P_1, \dots, P_k$ such that the endpoints of P_i are s_i and t_i. We prove every 6-connected graph on n vertices with $5n - 14$ edges is 3-linked. This is optimal, in that there exist 6-connected graphs on n vertices with $5n - 15$ edges that are not 3-linked for arbitrarily large values of n.

1 Introduction and Results

A graph is *k-linked* if for every set of $2k$ distinct vertices $\{s_1, \dots, s_k, t_1, \dots, t_k\}$ there exist disjoint paths $P_1, \dots, P_k$ such that the endpoints of P_i are s_i and t_i. A natural question is whether or not there exists a function $f(k)$ such that every $f(k)$-connected graph is k-linked. Larman and Mani [6] and Jung [3] first showed that such a function $f(k)$ exists by showing that the existence of a topological complete minor of size $3k$ and $2k$-connectivity suffice to make a graph k-linked. This result, along with an earlier result of Mader's that sufficiently high average degree forces a large topological minor [7] proved that such a function f above does exist. Robertson and Seymour [8] proved in their Graph Minor series that $2k$-connectivity and the existence of a K_{3k} minor suffices to make a graph k-linked. This, together with bounds on the extremal function for complete minors by Kostochka [5] and Thomason [15] showed that average degree $O(k\sqrt{\log k})$ implies the existence of a K_k minor, and consequently, that $f(k) = O(k\sqrt{\log k})$ suffices. Random graphs show that the extremal function for a K_{3k} minor is $\Omega(k\sqrt{\log k})$, and as a consequence, this bound for $f(k)$ could not be further improved by only taking advantage of a complete minors. Bollobás and Thomason [1] showed that the same effect can be achieved by replacing the K_{3k} minor with a sufficiently dense (noncomplete) minor, whose existence requires only $ck|V(G)|$ edges for a constant c. Thus they improved the bound on

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$f(k)$ to $22k$ by showing that a $2k$-connected graph with $11kn$ edges is k-linked. In [13], we show that this can be improved, and every $10k$-connected graph is k-linked. Independently of [13], Kawarabayashi, Kostochka and Yu [4] proved that every $12k$-connected graph is k-linked.

When attention is restricted to small values of k, stronger results are known. Jung partially characterized graphs G without disjoint paths linking $\{s_1, s_2, t_1, t_2\} \subseteq V(G)$ in [3]. Complete characterizations were later independently proven in [11, 12, 14], as well as efficient polynomial time algorithms developed [12]. An immediate consequence of the characterization is that $f(2) = 6$. For the case $k = 3$, Robertson and Seymour showed that a polynomial time algorithm exists (see [8], [9], and [10] for an outline of the argument), however the algorithm uses unreasonable constants, and therefore implementation is infeasible. Recent work by Chen et al. in [2] shows that the K_9 minor required by Robertson and Seymour's argument can be relaxed and a K_9^- minor will suffice. Along with bounds for the existence of such a minor, they improve the bound of $f(3)$ to 18.

In this paper, we prove the optimal edge bound for ensuring a graph is 3-linked.

Theorem 1.1 *Every 6-connected graph G on n vertices with $5n - 14$ edges is 3-linked.*

This bound is best possible because in [13] we exhibited, for every integer $l \geq 1$, a 6-connected graph on $n = 4(l + 1)$ vertices with $5n - 15$ edges that is not 3-linked. An immediate corollary of Theorem 1.1 is the following:

Corollary 1.2 *Every 10-connected graph is 3-linked.*

Thomassen conjectured [14] that every $(2k + 2)$-connected graph is k-linked. It has been observed that K_{3k-1} with k disjoint edges deleted gives a counterexample to this conjecture for $k \geq 4$. However, it is still conjectured for $k = 3$ that $f(3) = 8$.

Our proof of Theorem 1.1 is not exactly short. We wish we could find an easier proof, but there are some obstacles, partially explained in the next section, that seem to necessitate several tedious steps. Lemma 3.1, and even the weaker Lemma 3.3, have proven useful in graph structure theory. We hope that Theorem 1.1, a generalization of Lemma 3.3, will also be of some use. Even better would be an analogue of Lemma 3.1 for three disjoint paths, but that seems out of reach at the moment.

2 Definitions and Outline of Proof

For the purposes of this paper, all graphs will be simple. Edges will be considered as subsets of vertices of size two. For notation, the edge connecting vertices u and v will be written uv. If G is a graph and $e \in E(G)$, we denote by G/e the graph obtained from G by contracting e and deleting all resulting parallel edges. For notation, $N(v)$ will denote the neighborhood of v; that is, the set of vertices adjacent to the vertex v. We will denote by $\delta(G)$ the minimum degree in a graph G. Given a set of vertices $X \subseteq V(G)$, $\partial(X)$ is the subset of vertices in X with a neighbor in $V(G)-X$. If H is a subgraph of G we abbreviate $\partial(V(H))$ by $\partial(H)$. When we say (A,B) is a *separation* of a graph G, we mean that the union of A and B is the whole of the vertex set of G and every edge of G has both ends in A or B. The *order* of a separation (A,B) is $|A\cap B|$. A separation is *trivial* if either $A=V(G)$ or $B=V(G)$, and non-trivial otherwise. Given a set $X\subseteq V(G)$, a separation of the pair (G,X) is a separation (A,B) with $X\subseteq A$. We will use the notation $G[A]$ to indicate the subgraph of G induced by the set of vertices A. For $X\subseteq V(G)$, we define $\rho_G(X)$ (or $\rho(X)$ when the graph G is understood from the context) to be the number of edges with at least one endpoint in X. Given a path P in a graph and two vertices x and y in $V(P)$, we denote the subpath of P with ends x and y by xPy.

We will need the following definitions.

Definition A *linkage* is a graph $\mathcal{P}$ where every component of $\mathcal{P}$ is a path.

Given a linkage $\mathcal{P}$, we will use the standard notation $V(\mathcal{P})$ for the set of vertices and $E(\mathcal{P})$ for the set of edges. Sometimes we shall regard $\mathcal{P}$ as a set of its components and write $P\in\mathcal{P}$ to mean that the path P is a component of $\mathcal{P}$. If every member of $\mathcal{P}$ has one end in X and the other in Y, then we say that $\mathcal{P}$ is a linkage from X to Y. In that case, we designate, for each path $P\in\mathcal{P}$, its end in X as the *origin* and its end in Y as the *terminus* of P. If both ends belong to $X\cap Y$, we make an arbitrary choice.

Definition Let G be a graph, let $t\geq 1$ be an integer, and let $X\subseteq V(G)$. The pair (G,X) is *t-linked* if for all $k\leq t$ and distinct vertices $s_1,s_2,\dots s_k,t_1,\dots,t_k\in X$, there exists a linkage $\mathcal{P}$ from $\{s_1,\dots,s_k\}$ to $\{t_1,\dots,t_k\}$ such that

1. for every i, there exists $P\in\mathcal{P}$ such that the origin of P is s_i and the terminus of P is t_i, and
2. no component of $\mathcal{P}$ has an internal vertex in X.

We say that the pair (G,X) is *linked* if (G,X) is $\lfloor |X|/2\rfloor$-linked. A separation (A,B) in G is k-linked if $(G[B],A\cap B)$ is k-linked.

Given a set X of vertices, a *linkage problem* is a set of pairwise disjoint subsets of X of size 2. A linkage problem $\mathcal{L}=\{\{s_1,t_1\},\dots,\{s_k,t_k\}\}$ is *feasible* if there exists a linkage $\mathcal{P}$ such that for every $i=1,\dots,k$, there exists a component $P\in\mathcal{P}$ such that the ends of P are s_i and t_i. Such a linkage $\mathcal{P}$ ensuring that the linkage problem $\mathcal{L}$ is feasible is said to *solve* the linkage problem $\mathcal{L}$. Again, consider the linkage problem $\mathcal{L}=\{\{s_1,t_1\},\dots,\{s_k,t_k\}\}$ on a set X of vertices. Given a linkage $\mathcal{P}$ from X to some set X', label the vertices of X' such that path $P\in\mathcal{P}$ with end s_i or t_i in X has its other end s_i' or t_i', respectively in X'. Then the linkage $\mathcal{P}$ *induces* the linkage problem $\mathcal{L}'=\{\{s_1',t_1'\},\dots,\{s_k',t_k'\}\}$ on X'.

Definition Given G a graph, $X\subseteq V(G)$, and α,β two positive integers, (G,X) is *(α,β)-massed* if

(M1) $\rho(V(G)-X)\geq\alpha|V(G)-X|+\beta$, and

(M2) every separation (A, B) of order at most $|X| - 1$ with $X \subseteq A$ satisfies

$$\rho(B - A) \leq \alpha|B - A|.$$

The idea behind the definition of (α, β)-massed is that the graph has the specified number of edges outside the set X, and no significant portion of those edges are separated from X by a small cut set.

Let us outline our proof of Theorem 1.1 now. We use the method developed in [13]. For the sake of the inductive argument we replace 6-connectivity by the weaker condition (M2), and the requirement that G have at least $5|V(G)| - 14$ edges by the more-or-less equivalent condition (M1) for $\alpha = 5$ and $\beta = 4$. Thus we will prove the stronger result stated formally as Theorem 5.1, that if $X \subseteq V(G)$ has size six and the pair (G, X) is $(5, 4)$-massed, then it is linked.

Now let (G, X) be a minimal counterexample, and let e be an edge of G with neither end in X. Then the graph G/e is not $(5, 4)$-massed, for otherwise a linkage in G/e can be extended to one in G. Thus G/e fails to satisfy (M1) or (M2). If G/e fails to satisfy (M1), then e belongs to at least five triangles, and if every edge e has this property, then that is very useful. For then the neighborhood of every vertex has minimum degree at least five. It is not hard to show that actually $\rho(V(G) - X) = 5|V(G) - X| + 4$, and hence there is a vertex v of degree at most nine. Let $N = G[N(v) \cup \{v\}]$. Then N is a fairly dense graph on at most 10 vertices, and it is almost 3-linked. Now it is possible to find six disjoint paths from X to $V(N)$; let X' be their ends in $V(N)$. If only N were 3-linked, we could link the vertices of X' within N and we would be done. Unfortunately, N need not be 3-linked, but it is close. However, being only close requires a lot of additional work, because we need to find a different set of six disjoint paths from X to $V(N)$, linking in a different pattern, or something else equally good. What we mean here is formalized as conditions (C1) and (C2) in Section 4. The entire Section 4 is devoted to the proof of an auxiliary lemma (Lemma 4.4) that enables us to get around the fact that N need not be 3-linked.

So that is what we do when G/e fails to satisfy (M1) for every edge e, and so we may assume that G/e fails to satisfy (M2) for some e. Thus G/e has a separation of order at most five violating (M2), and hence G has a separation (A, B) of order at most six such that $X \subseteq A$ and $\rho(B - A) \geq 5|B - A| + 1$. If $\rho(B - A) \geq 5|B - A| + 4$, then we can apply induction to the graph G and set $A \cap B$ and complete the proof that way, but if $\rho(B - A) \leq 5|B - A| + 3$, then we have a problem. The graph $G[B]$ does not have enough edges for induction to go through, and yet it has too many edges for $B - A$ to be simply deleted. So what we do is we delete $B - A$ and *add* three carefully selected edges to make up for the loss. After modifying this idea a bit (we need to delete several such sets $B - A$, as it turns out, and add three edges *per* separation (A, B)) it is possible to show that the resulting graph G^* is $(5, 4)$-massed, and so it has the required linkage by induction. But why does this linkage extend to one in G? To make sure this will be possible we need to be extremely careful at selecting the edges we will add. This is done in Lemma 5.13, which is quite technical and whose proof occupies entire Section 6.

Here is how the paper is organized. In Section 3 we prove a lemma about extremal functions for linkages with two components. The lemma follows easily from the well-known characterization of 2-linked graphs. In Section 4 we prove Lemma 4.4, which gives a sufficient condition for replacing a set of six disjoint paths by a different set of disjoint paths with desirable properties. The main proof is presented in Section 5, except that a proof of Lemma 5.13 is deferred until Section 6. In the short Section 7 we recall an example from [13] showing that the bound in Theorem 1.1 is best possible.

3 Linking Two Pairs of Vertices

We begin by examining edge bounds to ensure that a pair (G, X) is 2-linked where $|X| = 6$. To achieve this, we will use the following lemma about the number of edges it takes to force the pair (G, X) to be linked, for a graph G and a set $X \subseteq V(G)$ when $|X| < 6$. This lemma is proven easily from the characterization of 2-linked graphs. As mentioned above, several researchers independently characterized such graphs (see [3, 11, 12, 14]). We use the formulation from [10].

Lemma 3.1 *[10] Let s_1, s_2, t_1, t_2 be distinct vertices of a graph G, such that no separation (A, B) of G of order ≤ 3 has $s_1, s_2, t_1, t_2 \in A \neq V(G)$. Then the following are equivalent:*

1. *there do not exist vertex-disjoint paths P_1, P_2 of G such that P_i links s_i and t_i for $i = 1, 2$*
2. *G can be drawn in a disc with s_1, s_2, t_1, t_2 on the boundary in order.*

As an easy corollary to the above lemma, we get the following:

Corollary 3.2 *Let G be a graph and $s_1, s_2, t_1, t_2 \in V(G)$. If there do not exist paths linking s_1, t_1 and s_2, t_2, then there exist subsets of vertices $A, B_1, \ldots, B_k$ for some k with the following properties:*

1. *Every edge $e \in E(G)$ either has both ends in A or in B_i for some $i \in \{1, \ldots, k\}$.*
2. *For every i, $|A \cap B_i| \leq 3$ and every $j \neq i$, $B_i \cap B_j \subseteq A$.*
3. *$s_1, s_2, t_1, t_2 \in A$ and $G[A]$ can be drawn in a disc with s_1, s_2, t_1, t_2 on the boundary in that order.*

We use the above corollary to prove the following lemma.

Lemma 3.3 *Let G be a graph and $X \subseteq V(G)$ of size at most 6. Let (G, X) be $(5, 1)$-massed. Then*

(i) if $|X| \leq 5$, (G, X) is linked,

(ii) either (G, X) is 2-linked or every pair of adjacent vertices in X have a common neighbor in $V(G) - X$, and

(iii) if (G, X) is $(5, 2)$-massed, then (G, X) is 2-linked.

Proof The graph $G[V(G)-X]$ must have some connected component with edges to every vertex of X, lest G have some separation violating the definition of $(5,1)$-massed. Thus we may assume there exist distinct vertices s_1, t_1, s_2, t_2 such that there do not exist two disjoint paths P_1 and P_2 with the ends of P_i being s_i and t_i, for otherwise, the lemma holds. Let $X' = \{s_1, t_1, s_2, t_2\}$ and $X'' = X - X'$. By Corollary 3.2 applied to $G - X''$, there exist subsets $A, B_1, \ldots, B_k$ of $V(G) - X''$, with the properties stipulated in Corollary 3.2.

Then $\rho_G(V(G)-X) = \rho_{G[A \cup X'']}(A - X) + \sum_{i=1}^{k} \rho_G(B_i - A)$. Since $(A \cup (\bigcup_{i \neq k} B_i), B_k)$ is a separation of order at most 3 in $G - X''$ for every k, we see that $(A \cup X'' \cup (\bigcup_{i \neq k} B_i), B_k \cup X'')$ is a separation of order at most $|X| - 1$ in G. Thus, $\rho_G(B_i - A) \leq 5|B_i - A|$ for every i. Moreover, since $G[A]$ is planar and has at least one face of size at least four, we see that $\rho_{G[A]}(A - X') \leq 3|A - X| + 1$ and consequently, $\rho_{G[A \cup X'']}(A - X) \leq (3 + |X''|)|A - X| + 1$. If $|X| \leq 5$, then $\rho_{G[A \cup X'']}(A - X) \leq 5|A - X|$, contrary to the fact that (G, X) is $(5,1)$-massed. This proves the lemma when $|X| \leq 5$; in particular, it proves (i). Thus we may assume that $|X| = 6$. We have $\rho_{G[A \cup X'']}(A - X) \leq 5|A - X| + 1$, and hence (G, X) is not $(5,2)$-massed. Thus (iii) holds. But (G, X) is $(5,1)$-massed, and so the inequalities above hold with equality. In particular, both vertices in X'' are adjacent every vertex of $V(G) - X$, the graph $G[A]$ is a triangulation except for exactly one face of size four (incident with s_1, s_2, t_1, t_2), and the pairs of vertices s_1, t_1 and s_2, t_2 are not adjacent. It follows that every pair of adjacent vertices in X have a common neighbor in $V(G) - X$, as desired by (ii). $\square$

4 Extremal Functions for Rerouting Paths

In this section, we focus on graphs where we are given a linkage with components $P_1, \ldots, P_6$ and we want to know how many edges the graph can have before we can find a different linkage $P'_1, \ldots P'_6$ in the graph satisfying various properties.

We are given the following setup: a graph G, a set X of six vertices and a fixed linkage problem $\mathcal{L}$ on X, and six disjoint paths from X to some set X'. We want to show that if the graph has enough edges, subject to the graph having a basic amount of connectivity, then either we can reroute the six paths to arrive in a distinct linkage problem on X', or we can actually find a path linking one pair of the linkage problem $\mathcal{L}$, and still find paths from the remaining four vertices of X to X'. This arises in a natural way when we are attempting to prove the edge bound necessary to force a graph to be 3-linked.

The following will be a common hypothesis of several definitions and lemmas, and therefore it seems worthwhile to give it a name.

Hypothesis H**:** Let G be a graph and $X, X' \subseteq V(G)$ two sets of size 6. Let $\mathcal{P} = \{P_1, \ldots, P_6\}$ be 6 disjoint induced paths where the ends of P_i are $x_i \in X$ and $x'_i \in X'$. Let $\mathcal{L}$ be the linkage problem $\{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$, and let $\mathcal{L}'$ be the linkage problem $\{\{x'_1, x'_4\}, \{x'_2, x'_5\}, \{x'_3, x'_6\}\}$.

Before proceeding, we prove a general lemma about desirable linkages from a fixed subgraph to the vertex set of another linkage.

Definition Let k be an integer and let $\mathcal{P}$ be a linkage with k components from X to X' in a graph G, where $|X| = |X'| = k$. Let the vertices of X and X' and the components $P_1, P_2, \ldots, P_k$ of $\mathcal{P}$ be numbered such that the ends of P_i are $x_i \in X$ and $x'_i \in X'$. Let H be a subgraph of G, and let $\mathcal{Q}$ be a linkage from $V(H)$ to $V(\mathcal{P})$. We say a vertex $v \in V(P_i)$ is *left $\mathcal{Q}$-extremal* if $v \in V(\mathcal{Q})$ and v is the only vertex of $x_i P_i v$ that belongs to $V(\mathcal{Q})$. Similarly, we say $v \in V(P_i)$ is *right $\mathcal{Q}$-extremal* if $v \in V(\mathcal{Q})$, and v is the only vertex of $v P_i x'_i$ that belongs to $V(\mathcal{Q})$. We say a vertex v is *$\mathcal{Q}$-extremal* if it is either left or right $\mathcal{Q}$-extremal. We say that a vertex $v \in V(P_i)$ is *$\mathcal{Q}$-sheltered* if P_i has a $\mathcal{Q}$-extremal vertex and v belongs to the subpath of P_i with ends the left and right $\mathcal{Q}$-extremal vertices.

We say that $\mathcal{Q}$ is an *H-comb* if

1. for each $Q \in \mathcal{Q}$, its origin is in $V(H)$ and its terminus is a $\mathcal{Q}$-extremal vertex,
2. every $\mathcal{Q}$-extremal vertex is the terminus of some component of $\mathcal{Q}$, and
3. if some vertex of $V(H) \cap V(P)$ for some $P \in \mathcal{P}$ is not the terminus of any path $Q \in \mathcal{Q}$ and it is not $\mathcal{Q}$-sheltered, then every path of $\mathcal{Q}$ has length zero and P includes the terminus of at most one path in $\mathcal{Q}$.

Lemma 4.1 *Let G be a graph, let $k, t \geq 1$ be integers, and let H be a subgraph of G. Let $X, X' \subseteq V(G)$ with $|X| = |X'| = k$ and let $\mathcal{P}$ be a linkage from X to X' with components $P_1, \ldots, P_k$ such that the ends of P_i are $x_i \in X$ and $x_i \in X'$. Then either there exists a separation (A, B) of order strictly less than t with $X \cup X' \subseteq A$ and $V(H) \subseteq B$, or there exists an H-comb with t components.*

Proof: Let there be no separation as stated in the lemma. By Menger's theorem, there exists a linkage from $V(H)$ to $X \cup X'$ with t components and no internal vertices in $V(H) \cup X \cup X'$. Let us choose such a linkage $\mathcal{Q}$ such that $E(\mathcal{Q}) - E(\mathcal{P})$ is minimal.

Let $Q_1, \ldots, Q_t$ be the components of $\mathcal{Q}$. For $j = 1, \ldots, t$, let q_j be the origin of Q_j and let $w_j \in V(Q_j) \cap V(\mathcal{P})$. Let Q'_j be defined as $q_j Q_j w_j$, and let $\mathcal{Q}'$ denote the linkage $Q'_1 \cup \cdots \cup Q'_t$. Let us pick $w_1, w_2, \ldots, w_t$ such that

(i) each w_i is $\mathcal{Q}'$-extremal, and

(ii) subject to (i), $|V(\mathcal{Q}')|$ is minimal.

Such a choice is possible because each terminus of a path in $\mathcal{Q}$ is $\mathcal{Q}$-extremal. We make the following claim.

Claim 4.2 *Each $\mathcal{Q}'$-extremal vertex is a terminus of a path in $\mathcal{Q}'$.*

Proof: Suppose to the contrary that there exists a $\mathcal{Q}'$-extremal vertex $w \in V(P_i) \cap V(Q'_j)$ for some $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, t\}$, and $w \neq w_j$. Then replacing Q'_j by $q_j Q_j w$ yields a linkage that contradicts (ii). $\square$

It immediately follows that $\mathcal{Q}'$ satisfies conditions 1. and 2. in the definition of H-comb.

To prove that $\mathcal{Q}'$ satisfies Condition 3. in the definition of H-comb, let $x \in V(H) \cap V(P_i)$ be not $\mathcal{Q}'$-sheltered. We may assume from symmetry that the path $x_i P_i x$ is disjoint from $\mathcal{Q}'$. We may also assume that x is the only vertex of $V(H)$ in $x_i P_i x$. Since $x \in V(H) - V(\mathcal{Q}')$ and no internal vertex of a component of $\mathcal{Q}$ belongs to H, we deduce that $x \notin V(\mathcal{Q})$. We claim that $x_i P_i x$ is disjoint from $\mathcal{Q}$. Assume otherwise, and let y be the vertex of $\mathcal{Q}$ in $x_i P_i x$ closest to x, and let j be the index such that $y \in V(Q_j)$, The choice of x implies that $y Q_j q_j$ includes an edge not in $E(\mathcal{P})$. Thus replacing Q_j by $x P_i y \cup y Q_j w$, where w is the terminus of Q_j, yields a linkage that contradicts the minimality of $\mathcal{Q}$.

Thus $x_i P_i x$ is disjoint from $\mathcal{Q}$. Then the path $x_i P_i x$ could have been chosen for the linkage $\mathcal{Q}$ in lieu of another path. By the minimality of $E(\mathcal{Q}) - E(\mathcal{P})$, we deduce that $\mathcal{Q}$ is a subgraph of $\mathcal{P}$. By (ii), each Q'_j has length zero, and since $x_i P_i x$ is disjoint from $\mathcal{Q}$, we see that P_i includes the terminus of at most one path in $\mathcal{Q}'$. Thus Condition 3. in the definition of H-comb holds. $\square$

We will be looking for conditions to ensure that a graph satisfying Hypothesis H also satisfies one of the following conditions:

(C1) There exist disjoint paths $Q, Q_1, \dots Q_4$ and an index $j \in \{1, 2, 3\}$ such that Q links x_j and x_{j+3} and each $Q_1, \dots, Q_4$ has an end in X and the other end in X'.

(C2) There exist disjoint paths $Q_1, \dots, Q_6$ with the ends of Q_i being x_i and q_i, where $q_i \in X'$ for all i. Furthermore, the linkage problem $\{\{q_1, q_4\}, \{q_2, q_5\}, \{q_3, q_6\}\}$ is distinct from $\mathcal{L}'$.

We define

Definition Let $X \subseteq V(G)$. We will say a separation (A, B) of G is a *rigid separation* of (G, X) if $X \subseteq A$, $B - A \neq \emptyset$ and $(G[B], A \cap B)$ is linked.

Let us recall that, for a subgraph H of a graph G, $\partial(H)$ is the set of all vertices of H that have a neighbor in $V(G) - V(H)$.

Lemma 4.3 *Let G be a graph satisfying Hypothesis H. Let H be an induced subgraph with $|\partial(H)| \geq 5$ and further, assume the following conditions hold:*

1. *At most two of the paths in $\mathcal{P}$ intersect H in more than one vertex, and at most 3 paths total intersect $V(H)$.*

2. *For any distinct vertices $v, s_1, s_2, t_1, t_2 \in \partial(H)$, there exist paths Q_1, Q_2 with ends s_1, t_1 and s_2, t_2 respectively with all internal vertices of the paths in $V(H) - v$.*

Then either (C1) or (C2) holds, or the pair $(G, X \cup X')$ has a rigid separation of order at most four.

While technical, this lemma is saying something fairly intuitive. In Hypothesis H, we are given the six paths in G, and some subgraph H that allows us to cross paths that enter H. By Lemma 4.1, if there does not exist an H-comb with five components, then there exists a small separation separating $X \cup X'$ from H which will necessarily be rigid. Otherwise, we find such an H-comb. Then the H-comb either allows us to cross two of the paths to arrive at X' in a distinct linkage problem, or we can link one pair of terminals in the linkage problem $\mathcal{L}$ and still link the other four vertices in X to X'.

Proof: Assume the statement is false, and let G be as in Hypothesis H forming a counterexample.

If there exists a separation of order at most 4 separating $X \cup X'$ from $V(H)$, then by the assumptions on H, the separation must be rigid. Thus no such separation exists, and by Lemma 4.1, there exists an H-comb of $\mathcal{Q}$ with five components. Let the components of $\mathcal{Q}$ be labeled $Q_1, \ldots, Q_5$. Let q_i be the origin of Q_i in H.

We claim that every vertex of $V(H) \cap V(\mathcal{P})$ is $\mathcal{Q}$-sheltered. To see that, let $x \in V(H) \cap V(\mathcal{P})$, and suppose for a contradiction that x is not $\mathcal{Q}$-sheltered. By property 3. in the definition of comb, every path Q_i is trivial and hence at least three paths in $\mathcal{P}$ intersect H, with each intersection corresponding to a trivial path in $\mathcal{Q}$. Then by our assumptions, exactly three paths in $\mathcal{P}$ do, two in at least 2 vertices say P_i and P_j, and one in exactly one vertex, say P_k. By 3. in the definition of H-comb, x cannot lie on P_i or P_j. As a result, either $x \in P_k$ and three paths of $\mathcal{P}$ intersect H in at least 2 vertices, or there is a fourth path of $\mathcal{P}$ intersecting H. Either case is a contradiction to our assumptions. Hence every vertex in $V(H) \cap V(\mathcal{P})$ is $\mathcal{Q}$-sheltered.

Because the five termini of $\mathcal{Q}$ are distributed among the 6 paths of $\mathcal{P}$, there are two cases to consider.

Case 1: *There exists an index i such that P_i and P_{i+3} both include a terminus of a path in $\mathcal{Q}$.*

Without loss of generality, assume that P_1 contains the terminus y_1 of Q_1 and P_4 contains the terminus y_2 of Q_2. Then there is at most one other path containing 2 termini of $\mathcal{Q}$. As a subcase, assume some P_j, $j \neq 1, 4$ contains two termini of $\mathcal{Q}$. Without loss of generality, let P_2 has y_2 the terminus of Q_3 and z_2 the terminus of Q_4. Then there exist disjoint paths R_1, R_2 in H where R_1 links q_1 and q_2 and R_2 links q_3 and q_4. We can pick R_1 and R_2 to avoid q_5, and so by the previous paragraph, we see that R_1 and R_2 have no internal vertices in $V(\mathcal{P}) - (V(y_1 P_1 z_1) \cup V(y_2 P_2 z_2))$. Then the linkage

$$x_1 P_1 y_1 Q_1 q_1 R_1 q_2 Q_2 y_4 P_4 x_4, \quad x_2 P_2 y_2 Q_3 q_3 R_2 q_4 Q_4 z_2 P_2 x_2', \quad P_3, P_5, P_6$$

satisfies (C1).

Otherwise, each Q_i, $i \geq 3$ has its terminus in a different path of $\mathcal{P}$. Then each of P_2, P_3, P_5, and P_6 have at most one vertex in $V(H)$, and any such vertex in H must be equal to q_i for some i. By our assumptions on H, there exists a path R in H linking q_1 and q_2 avoiding q_3, q_4, and q_5. The paths

$$x_1 P_1 y_1 Q_1 q_1 R q_2 R_2 y_4 P_4 x_4, \quad P_2, P_3, P_5, P_6$$

satisfy (C1).

Case 2: *There exist indices i and $j \neq i, i+3$ such that P_i and P_j each contain at least two termini of $\mathcal{Q}$*

Without loss of generality, let P_1 contain the terminus y_1 of Q_1 and the terminus z_1 of Q_2. Let P_2 contain the terminus y_2 of Q_3 and the terminus z_2 of Q_4. Observe that q_5 is the only possible vertex of P_3, P_4, P_5, P_6 to lie in $V(H)$. By our assumptions on H, H contains disjoint paths R_1 linking q_1 and q_4 and R_2 linking q_2 and q_3 avoiding the vertex q_5. Then the linkage

$$x_1P_1y_1Q_1q_1R_1q_4Q_4z_2P_2x_2' \quad x_2P_2y_2Q_3q_3R_2q_2Q_2z_1P_1x_1', \quad P_3, P_4, P_5, P_6$$

satisfies (C2). This completes the proof of the lemma.□

Now we immediately apply the previous lemma in proving the following result about the necessary number of edges in a graph to guarantee (C1) or (C2).

Lemma 4.4 *Let G be a graph satisfying Hypothesis H. If*

1. *$\rho(V(G) - X) \geq 5|V(G) - X| + 1$, and*
2. *every separation (A, B) of order at most 4 with $X, X' \subseteq A$ satisfies*

$$\rho(B - A) \leq 5|B - A|,$$

then G satisfies (C1) or (C2).

Proof: Assume the lemma is false, and let G be a counterexample satisfying Hypothesis H on a minimal number of vertices, and, subject to that, with $\rho(V(G) - X)$ minimal. We assume that X has an edge between all possible pairs of vertices of X except for the pairs $(x_1, x_4), (x_2, x_5), (x_3, x_6)$. Adding these edges if necessary clearly does not change the truth or falsehood of the hypotheses or conclusions of the lemma. We proceed in a series of claims, some of them borrowed from [13]. We include proofs for the sake of completeness.

Claim 4.5 *$(G, X \cup X')$ has no rigid separation of order at most four.*

Proof: Let (A, B) be such a separation, and assume we have chosen it to maximize $|B|$. Consider the graph G' that is defined to be the graph obtained from $G[A]$ by adding edges between every pair of non-adjacent vertices in $A \cap B$. For notation, let $S := A \cap B$. By Condition 2 in the statement of the lemma, it follows that $\rho(V(G') - X) \geq 5|V(G') - X| + 1$. Also, we know that G' has six disjoint paths from X to X' with the same path ends as in G since any path in G that uses vertices of $B - A$ can be converted to a path in G' because $G'[S]$ is complete. Let the paths be labeled $P_1', \ldots, P_6'$ with the ends of P_i' being x_i and x_i'. For paths in G' satisfying (C1) or (C2), we may assume that each path uses at most one edge of $G'[S]$. Because edges in S

may be extended to paths in G with all internal vertices in $B - A$, we know any paths in G' satisfying (C1) or (C2) extend to paths in G. If G' satisfies Condition 2 in the statement of the lemma, by minimality it follows that G also satisfies (C1) or (C2), a contradiction. Thus we see that G' has a separation violating Condition 2. Let (A', B') be such a separation, and assume (A', B') is chosen to minimize $|B'|$. Then by Lemma 3.3, it follows that $(G'[B'], A' \cap B')$ is linked. Because $G'[S]$ is complete, we know that $S \subseteq A'$ or B'. If $S \subseteq A'$, then $(A' \cup B, B')$ is a separation in G violating Condition 2 in the statement of the lemma. Consequently $S \subseteq B'$. As we saw above, disjoint paths in G' linking terminals in $A' \cap B'$ extend to disjoint paths in G, and hence, $(A', B' \cup B)$ is a rigid separation in G violating our choice of (A, B).

This contradiction completes the proof that $(G, X \cup X')$ has no rigid separation of order at most four. $\square$

Claim 4.6 *G has no nontrivial separation (A, B) of order six with $X \subseteq A$ and $X' \subseteq B$.*

Proof: Assume otherwise, and let (A, B) be such a separation. Then if we consider $G[A]$, X, and $A \cap B$, the linkage $\mathcal{P}$ induces a natural labeling of $A \cap B = \{a_1, \ldots, a_6\}$ with $\{a_i\} = V(P_i) \cap A \cap B$. The graph $G[A]$, the sets X, $A \cap B$, and the paths of $\mathcal{P}$ restricted to A satisfy Hypothesis H. Similarly, $G[B]$, $A \cap B$, and X' also satisfy Hypothesis H. Condition 2 will naturally hold in $G[A]$ and $G[B]$. Moreover, in at least one of $G[A]$ or $G[B]$, Condition 1 will also hold. By the minimality of G as a counterexample, one of $G[A]$ or $G[B]$ has paths satisfying (C1) or (C2), and consequently, G would as well. This contradiction proves the claim. $\square$

Now we attempt to contract an edge e, $e \not\subseteq E(G[X \cup X']) - E(\bigcup_i P_i)$. This may have the effect of merging two vertices, x_j and x'_j into a single vertex, which we will consider to be a member of both X and X' in G/e connected by a path of length zero. Since G has no nontrivial separation of order six separating X from X', we know that G/e has six paths $P_1^*, \ldots, P_6^*$ from X to X'. Let the ends of P_i^* be x_i and y'_i. If the linkage problems $\{\{x'_1, x'_4\}, \{x'_2, x'_5\}, \{x'_3, x'_6\}\}$ and $\{\{y'_1, y'_4\}, \{y'_2, y'_5\}, \{y'_3, y'_6\}\}$ are distinct, then the paths P_i^* in G/e extend to disjoint paths P'_i with the same endpoints in G satisfying (C2). This implies that $\{\{x'_1, x'_4\}, \{x'_2, x'_5\}, \{x'_3, x'_6\}\} = \{\{y'_1, y'_4\}, \{y'_2, y'_5\}, \{y'_3, y'_6\}\}$, and so for the sake of this paragraph we may assume that by possibly renumbering the vertices of X', the ends of P_i^* are x'_i and x_i. If G/e were to satisfy Conditions 1. and 2. in the statement, then by the minimality of G, G/e has paths $Q_1^*, \ldots Q_k^*$ satisfying (C1) or (C2). Those paths extend to paths $Q'_1, \ldots Q'_k$ in G satisfying (C1) or (C2). Thus we have proven contracting the edge e violates one of the hypotheses of the lemma.

Claim 4.7 *G/e violates Condition 1. for every edge $e \not\subseteq X$, $e \not\subseteq X'$.*

Proof: We have seen above that G/e must violate Condition 1. or 2. Assume to reach a contradiction, that G/e has a separation (A', B') violating Condition 2. Pick such a separation to minimize the size of B'. Let v_e be the vertex of G/e corresponding to the contracted edge e, and let P_i^*, P'_i be as in the previous paragraph. Then if $v_e \in A' - B'$, the separation (A', B') induces a separation (A, B) in G violating Condition 2 in the statement of the lemma. We conclude that $v_e \in B'$. By Lemma 3.3, $(G/e[B'], A' \cap B')$ is linked. If $v_e \in B' - A'$, (A', B') induces a rigid separation in G of order at most four, contrary to Claim 4.5. Thus we may assume in fact that $v_e \in A' \cap B'$, and (A', B') induces a separation (A, B) of order five in G with $X, X' \subseteq A$ and $\rho(B - A) \geq 5|B - A| + 1$. Also, since only one path of the P_i' uses endpoints of e, we know that at most four paths of the P_i' use vertices of B. If exactly four of the paths P_i' use vertices of B, then there exists an index $i = 1, 2, 3$ such that P_i' and P_{i+3}' both use vertices of B. Without loss of generality, assume P_1' and P_4' use vertices of B. It follows that no path can use vertices of $B - A$. The graph $G[B - A]$ must have some connected component with all of $A \cap B$ as neighbors, since $\rho(B - A) \geq 5|B - A| + 1$. Then the pair of terminals x_1 and x_4 can be connected with a path using vertices of B without intersecting the remaining paths P_2', P_3', P_5', P_6' and consequently G satisfies (C1). Thus we may assume that at most three of the paths P_i' use vertices of B, and because $|A \cap B| = 5$, at most two paths use more than one vertex of B. By Lemma 3.3, $(G[B], A \cap B)$ is linked. We have shown $G[B]$ satisfies all the conditions of Lemma 4.3. Since G has no rigid separation of order at most four, we then know that G would satisfy (C1) or (C2), a contradiction. $\square$

Thus we may assume that contracting the edge e violates Condition 1. in the statement of the lemma. We will show that the endpoints of e have five common neighbors. We refer to these common neighbors as *triangles* containing e. We prove

Claim 4.8 *Every edge $e \nsubseteq X$, $e \nsubseteq X'$ is contained in at least five triangles.*

Proof: Given such an edge e, by Claim 4.7 we see that G/e violates Condition 1 in the statement of the lemma. Since G/e has exactly one fewer vertex in $G/e - X$, the edge count must decrease by at least five. If $e \cap X = \varnothing$, then the decrease in the edge count corresponds to the number of common neighbors of u and v. Thus the endpoints of e have at least five common neighbors, proving the claim. If $e = uv$ and $v \in X$, then upon contracting e, the edge count decreases by the sum the number of common neighbors of u and v and the number of neighbors of u in X besides v. Without loss of generality, assume that $v = x_1$. We know that u is not adjacent to x_4, since by Claim 4.6 there exist four paths from $X - \{x_1, x_4\}$ to $X' - \{x_1', x_4'\}$ not containing the vertex u. Moreover, we have already assumed that x_1 is adjacent to all vertices of X besides x_4. Then, in fact, all the neighbors of u in X are common neighbors with v, and u and v consequently have at least five common neighbors. $\square$

Similarly to when we contracted an edge, if $e \nsubseteq X$ and if $G - e$ satisfies the conditions of the lemma, then by minimality, there exist paths in $G - e$ satisfying (C1) or (C2). Those paths would also exist in G. We conclude that $G - e$ violates Condition 1. or 2. of the lemma.

Claim 4.9 *For any edge $e \nsubseteq X$, $e \nsubseteq X'$, $G - e$ violates Condition* 1.

Proof: Assume to reach a contradiction that there exists a separation (A, B) of $G - e$ violating Condition 2. in the lemma. In order for (A, B) not to induce a separation in G violating Condition 2 in the statement of the lemma, it must be the case that one end of e belongs to $A - B$ and the other end to $B - A$. But the ends of e must have at least five neighbors in common, and all these common neighbors must lie in $A \cap B$. This contradicts the order of (A, B), proving the claim. $\square$

Since $G - e$ does not satisfy Condition 1 in the statement of the lemma, as an immediate consequence we see:

Claim 4.10 $\rho(V(G) - X) = 5|V(G) - X| + 1$.

We now show that we can find a vertex of small degree outside the sets X and X'. Let

$$A := V(G) - X - X'.$$

First, we see that A is not empty.

Claim 4.11 $V(G) \neq \bigcup_{i=1}^{6} V(P_i)$.

Proof: Assume that $V(G)$ does in fact consist of the vertices of the paths $P_1, \ldots, P_6$. Some path must be non-trivial, since it is not the case that $X = X' = V(G)$. Without loss of generality, assume P_1 is non-trivial, and let uv be an edge on P_1, with $x_1 u, v, x_1'$ occurring on P_1 in the order listed. We may also assume no vertex of P_1 has a neighbor on P_4, lest we satisfy (C1). We see that u and v have five common neighbors on the paths P_2, P_3, P_5, P_6. Then u and v have two common neighbors on the same path, say P_2, call them r and s, and assume r precedes s on the path P_2. Then we get paths

$$x_1 P_1 u s P_2 x_2', \quad x_2 P_2 r v P_1 x_1', \quad P_3, \ldots, P_6$$

satisfying (C2), proving the claim. $\square$

First we prove two facts we will use repeatedly in analyzing the cases to come is the following:

Claim 4.12 *G contains no K_5 subgraph.*

Proof: The statement follows immediately from Lemma 4.3 and the fact that G has no rigid separation of order at most four. $\square$

Claim 4.13 *For any vertex $v \in A$, there exist six disjoint paths $P_1^*, \ldots, P_6^*$ in G where the ends of P_i^* are $x_i \in X$ and $y_i \in X'$ such that the paths avoid v and the linkage problem $\{\{y_1, y_4\}, \{y_2, y_5\}, \{y_3, y_6\}\}$ is equal to $\mathcal{L}$.*

Proof: Given such a vertex $v \in A$, by Claim 4.6, we know there exist $P_1^*, \dots, P_6^*$ such that the ends of P_i^* are $x_i \in X$ and $y_i \in X'$. To see this, consider $G - v$. If there did not exist six disjoint paths from X to X', then $G - v$ would contain a separation (A, B) of order at most five with $X \subseteq A$ and $X' \subseteq B$. Then $(A \cup \{v\}, B \cup \{v\})$ is a nontrivial separation in G separating X from X' of order at most six, a contradiction to Claim 4.6.

If the paths $P_1^*, \dots, P_6^*$ induced a distinct linkage problem on X', this would violate our choice of G as a counterexample. $\square$

The next claims establish that there exists a vertex in A of small degree.

Claim 4.14 *Every vertex in A has at most six neighbors in $X \cup X'$.*

Proof: Assume $v \in A$ has strictly more than six neighbors in $X \cup X'$. By Claim 4.13, we may assume $v \notin \bigcup_i V(P_i)$. Then there exists some index $i \in \{1, 2, 3\}$ such that v has both x_i and x_{i+3} as neighbors, or v has both x_i' and x_{i+3}' as neighbors. Then we are able to link x_i and x_{i+3} through the vertex v and still find paths from the remaining four vertices of X to X'. The graph G would then satisfy (C1), a contradiction. $\square$

Claim 4.15 *There exists a vertex in A of degree at most* 11.

Proof: Assume otherwise. If we let $f(x)$ be the number of neighbors a vertex $x \in X$ has in $V(G) - X$, then we see

$$2\rho(V(G) - X) = \sum_{v \in A} \deg(v) + \sum_{x \in X' - X} \deg(x) + \sum_{x \in X} f(x)$$

By assumption, every vertex in A has degree at least 12, and every vertex $v \in X' - X$ has some neighbor u on the path P_i terminating at v. As we saw above, the edge uv is in at least five triangles, implying that v has degree at least six. Thus we see

$$2\rho(V(G) - X) \geq 12|A| + 6|X' - X| + \sum_{x \in X - X'} f(x)$$

Each vertex $v \in X - X'$ has some neighbor u on the path P_i beginning at v, and the edge uv is in at least five triangles. Since we know that v has at most four neighbors in X, $f(v) \geq 2$. Thus

$$\begin{aligned}
2\rho(V(G) - X) &\geq 12|A| + 6|X' - X| + 2|X - X'| \\
&= 10|V(G) - X| + 2|V(G) - X| - 4|X' - X| \\
&= 10|V(G) - X| + 2|A| - 2|X' - X|.
\end{aligned}$$

Then because vertices in A have at most six neighbors in $X \cup X'$, we know that $|A| \geq 7$. But in fact, if $|A| = 7$, $G[A] = K_7$, contradicting the fact that G has no K_5 subgraph. Thus we may assume that $|A| \geq 8$. The above equation then contradicts the fact that $\rho(V(G) - X) = 5|V(G) - X| + 1$. $\square$

Let $v \in A$ be a vertex of degree at most 11. We show that the neighborhood of v, $N(v)$, is sufficiently dense to apply Lemma 4.3. By Claim 4.8, we see that that the minimum degree of $G[N(v)]$ is five. By Claim 4.13, we may assume that none of the paths $P_1, \ldots, P_6$ uses the vertex v.

Claim 4.16 *There does not exist an index $i \in \{1, 2, 3\}$ such that P_i and P_{i+3} both intersect $N(v)$.*

Proof: Assume otherwise, and without loss of generality, that P_1 and P_4 intersect $N(v)$. Then P_1 can be linked to P_4 using the vertex v, implying that G would satisfy (C1), a contradiction. $\square$

Claim 4.17 *At most three paths of $\mathcal{P}$ intersect $N(v)$. If exactly three such paths do intersect $N(v)$, then one of them contains exactly one vertex of $N(v)$.*

Proof: If four or more paths use vertices of $N(v)$, then there exists an index i such that P_i and P_{i+3} both intersect $N(v)$, contradicting Claim 4.16. Assume exactly three paths do, and further assume that all three paths use at least two vertices of $N(v)$. Again by Claim 4.16, we may assume the three paths are P_1,P_2, and P_3.

Let $S := N(v) \cap V(\mathcal{P})$ and $T = N(v) - V(\mathcal{P})$. Let s_i and t_i be the first and last vertex of S on P_i, respectively. Then $|S| \geq 6$ and hence $|T| \leq 5$. We claim that for distinct integers $i, j \in \{1, 2, 3\}$

($\star$) There is no path Q from $s \in S \cap V(P_i) - \{t_i\}$ to $t \in S \cap V(P_i) - \{s_j\}$ with interior in T.

Indeed, if such a path Q exists, say for $i = 1$ and $j = 2$, then the paths

$$x_1P_1sQtP_2x_2', \quad x_2P_2s_2vt_1P_1x_1', \quad P_3, P_4, P_5, P_6$$

satisfy (C2), a contradiction.

In particular, ($\star$) implies that every $s \in S$ has at most three neighbors in S because s_1 has at most one neighbor in $V(P_1) \cap S$ (since P_1 is induced) and at most two in $S - V(P_1)$, namely s_2 and s_3. If $s \in V(P_1) - \{s_1, t_1\}$, then s has at most two neighbors in $V(P_1) \cap S$ and none in $S - V(P_1)$. Thus each $s \in S$ has at least two neighbors in T by Claim 4.8. Also by ($\star$), the neighbors in T of the vertices s_1 and t_2 belong to different components of $G[T]$; thus, in particular, $G[T]$ has at least two components and $|T| \geq 4$. Hence $|S| \leq 7$. Since $|T| \leq 5$, some component of $G[T]$, say J, has at most two vertices. By ($\star$) the neighbors of J that belong to S are contained in one of the following sets: $S \cap V(P_1)$, $S \cap V(P_2)$, $S \cap V(P_3)$, $\{s_1, s_2, s_3\}$, or $\{t_1, t_2, t_3\}$. Since $|S| \leq 7$, each of these sets has at most three vertices. Yet each vertex of J has at least five neighbors in $S \cup T$ by Claim 4.8, a contradiction. $\square$

In order to apply Lemma 4.3 and complete the proof of the lemma, all that remains to show is the following claim.

Claim 4.18 *Let $S := \{s_1, t_1, s_2, t_2, x\}$ be vertices in $N(v)$. There exist paths in $G[N(v) \cup v]$ linking s_1 to t_1 and s_2 to t_2 that do not contain the vertex x.*

Proof: Consider the vertices s_1 and t_1. We may assume that s_1 is not adjacent to t_1, for otherwise the paths s_1t_1 and s_2vt_2 would satisfy the claim. Each of s_1 and t_1 must have at least two neighbors each in $N(v) - S$ by Claim 4.8. We may assume these neighbors are in different components of $G[N(v) - S]$, otherwise, connect s_1 and t_1 with a path in $N(v) - S$ and link s_2 and t_2 with the vertex v. But each vertex in $N(v) - S$ can have at most three neighbors in S, lest s_i and t_i have a common neighbor for one of the values of i. Thus each vertex of $N(v) - S$ has at least two neighbors in $N(v) - S$. Since $G[N(v) - S]$ must have at least two connected components, we see that it in fact consists of two disjoint K_3 subgraphs and every vertex in $N(v) - S$ has exactly three neighbors in S. But then every vertex in $N(v) - S$ is adjacent to x. Then one of the K_3 subgraphs in $N(v) - S$ along with x and v forms a K_5 subgraph in G, a contradiction to Claim 4.12. Thus, in fact we are able to link the pairs (s_i, t_i) and avoid the vertex x. □

We have shown that the subgraph $G[N(v) \cup v]$ satisfies all the requirements of Lemma 4.3. Because we have shown in Claim 4.5 that (G, X) does not have any rigid separations of order at most four, we arrive at the final contradiction to our choice of G to not satisfy (C1) or (C2). □

5 The Extremal Function for 3-linkages

For the proof of Theorem 1.1, we consider the following stronger statement.

Theorem 5.1 *Given a graph G and $X \subseteq G$ with $|X| = 6$, if (G, X) is $(5, 4)$-massed, then it is linked.*

We first see that Theorem 1.1 follows easily from Theorem 5.1.

Proof of Theorem 1.1, assuming Theorem 5.1 Let G be 6-connected, with $|E(G)| \geq 5|V(G)| - 14$. Fix a set X of six vertices and a linkage problem $\mathcal{L}$ on X. Label the vertices of X such that $\mathcal{L} = \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$. Let t be the number of edges of G with both ends in X. Then

$$\rho(V(G) - X) = |E(G)| - t \geq 5|V(G)| - 14 - t = 5|V(G) - X| + 16 - t.$$

If $t = 15$, then the linkage problem $\mathcal{L}$ is feasible because $G[X]$ is a clique. If $t = 13$ or $t = 14$, then x_i is adjacent to x_{i+3} for at least one index $i \in \{1, 2, 3\}$, and hence $\mathcal{L}$ is feasible by Lemma 3.3 (*iii*). Finally, if $t \leq 12$, then $\mathcal{L}$ is feasible by Theorem 5.1. □

In the rest of this section we prove Theorem 5.1, modulo the technical Lemma 5.13, whose proof we delegate to the next section. The proof method is again inspired by [13]. To begin the proof we rigorously define what we mean by a minimal counterexample to Theorem 5.1.

Definition Let G be a graph, $X \subseteq V(G)$ with $|X| = 6$, and let $\mathcal{L}$ be a linkage problem on X. Assume the vertices of X are labeled such that $\mathcal{L} = \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$. Then the triple $(G, X, \mathcal{L})$ is 3-*minimal* if the following hold:

(A) (G, X) is $(5,4)$-massed.

(B) The linkage problem $\mathcal{L}$ is not feasible.

(C) Subject to (A) and (B), $|V(G)|$ is minimal.

(D) Subject to (A), (B) and (C), $\rho(V(G) - X)$ is minimal.

(E) Subject to (A), (B), (C) and (D), the number of edges of $G[X]$ is maximal.

In the course of the proof, we will ensure that every edge of a 3-minimal triple $(G, X, \mathcal{L})$ not contained in X is in five triangles. This means the neighborhood of a vertex $v \in V(G) - X$ will induce a subgraph N of minimum degree five. Moreover, we will see in Claim 5.16 that the edge bound (M1) in the definition of $(5,4)$-massed is satisfied with equality. Since the graph then has strictly less than $5|V(G)|$ edges, we know G has a vertex of degree at most nine. Additionally we show that there exists such a vertex v of degree at most nine not contained in the set X. We attempt to find disjoint paths from X to the neighborhood of v. The graph $G[N(v) \cup \{v\}]$ is sufficiently dense that if we consider a set X' of at most five vertices in $N(v)$, the pair $(G[N(v) \cup \{v\}], X')$ is 2-linked. Thus if there exists a small separation separating X from $N(v)$ in the graph G, the pair (G, X) will have a rigid separation. The existence of a rigid separation will provide a contradiction to our choice of a 3-minimal triple $(G, X, \mathcal{L})$.

Given that no small separation exists, by Menger's Theorem there exist six disjoint paths from X to $N(v)$. If we let X' be the set of ends of the paths in $N(v)$ then the linkage problem on X naturally gives a linkage problem $\mathcal{L}'$ on the path ends X'. Let N be the subgraph induced by $N(v)$. Ideally, we would link two pairs of $\mathcal{L}'$ in the subgraph N, and link the third pair of terminals using the vertex v. It is not the case, however, that any such two pairs of the linkage problem $\mathcal{L}'$ can be linked in N. This leads us to the following definition.

Definition Let G be a graph, $X \subseteq V(G)$ with $|X| = 6$, and let $\mathcal{L}$ be a linkage problem on X consisting of three pairs of vertices. Let the vertices of X be labeled such that $\mathcal{L} = \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$. The triple $(G, X, \mathcal{L})$ is *quasi-firm* if there exist distinct indices i and j in $\{1, 2, 3\}$ and disjoint paths P_i and P_j with all internal vertices in $V(G) - X$ with the ends of P_i equal to x_i and x_{i+3} and the ends of P_j equal to x_j and x_{j+3}.

In our 3-minimal triple $(G, X, \mathcal{L})$ above, we do not need (N, X') to be 2-linked and that we be able to link any two pairs of vertices; it would suffice that only some two pairs of vertices in $\mathcal{L}$ could be linked. If the triple $(N, X', \mathcal{L}')$ were quasi-firm, we could link the final pair of vertices of $\mathcal{L}'$ using the vertex v adjacent all of $N(v)$. Unfortunately, it is not the case that $(N, X', \mathcal{L}')$ will always be quasi-firm, but the instances where it is not are limited in scope.

Following the strategy of [13] we prove that a 3-minimal triple cannot contain a rigid separation of order at most six.

Lemma 5.2 *Let $(G, X, \mathcal{L})$ be a 3-minimal triple. Then the pair (G, X) does not have a nontrivial rigid separation of order at most 6.*

Proof: Assume that (G, X) does have a nontrivial rigid separation, call it (A, B). Assume from all such rigid separations, we pick (A, B) such that $|A|$ is minimized. If $|A \cap B| = 6$, then find six paths from X to $A \cap B$ in $G[A]$. If such paths existed, we could link the endpoints of the paths as prescribed by the linkage problem $\mathcal{L}$ given the fact that $(G[B], A \cap B)$ is linked. But if those six paths did not exist, then $G[A]$ contains a separation (A', B') of order less than six with $X \subseteq A'$ and $A \cap B \subseteq B'$. But such a separation (A', B') chosen of minimal order induces a rigid separation of (G, X), namely $(A', B \cup B')$, violating our choice of (A, B).

Now assume (A, B) has order at most five. Let G' be obtained from G in the following manner. The graph G' is equal to $G[A]$ with additional edges added to every non-adjacent pair of vertices in $A \cap B$. Thus $G'[A \cap B]$ is a complete subgraph. By (M2) we deleted at most $5|B - A|$ edges when we deleted the vertices of $B - A$, and as a result, $\rho(V(G') - X) \geq 5|A - X| + 4$. Assume that (G', X) also satisfies condition (M2). Then by (C), we know linkage problem $\mathcal{L}$ is feasible in G'. Take three paths linking the pairs of $\mathcal{L}$, and choose them to be as short as possible. Then each path uses at most one edge in $A \cap B$ because $G'[A \cap B]$ is a complete subgraph. These disjoint edges can be extended to disjoint paths in G with every internal vertex in $B - A$ by the fact that $(G[B], A \cap B)$ is linked. Thus the linkage problem $\mathcal{L}$ would be feasible in G, a contradiction.

Consequently, the pair (G', X) has a separation violating (M2). Let (A', B') be such a separation, and assume it is picked such that $|B'|$ is minimized. Because $G'[A \cap B]$ is a complete subgraph, $A \cap B \subseteq A'$ or $A \cap B \subseteq B'$. If $A \cap B \subseteq A'$, then $(A' \cup B, B')$ would be a separation in G violating (M2). Thus $A \cap B \subseteq B'$. Given our choice of (A', B'), we know $(G'[B'], A' \cap B')$ is $(5, 1)$-massed. By Lemma 3.3 (i), we know that $(G'[B'], A' \cap B')$ is linked. Disjoint paths in $G'[B']$ using edges of $A \cap B$ can be extended as in the previous paragraph, so we see that $(A', B' \cup B)$ is a rigid separation of G, violating our choice of (A, B). This proves the lemma. $\square$

The following lemma will be used to show that if $(G, X, \mathcal{L})$ is a 3-minimal triple, $v \in V(G) - X$ has degree at most nine, and $X' \subseteq N(v)$ satisfies $|X'| \leq 5$, then $(G[N(v) \cup \{v\}], X')$ is 2-linked.

Lemma 5.3 *Let G be a graph and $X \subseteq V(G)$ with $|X| \leq 5$. Assume that $\delta(G) \geq 6$, $|V(G)| \leq 10$, and moreover, assume there exists a vertex $v \in V(G) - X$ adjacent to every other vertex of G. Then (G, X) is 2-linked.*

Proof: Let $\mathcal{L}$ be a linkage problem on X. Clearly, we may assume that $|X| \geq 4$ and $\mathcal{L}$ consists of two pairs of vertices, otherwise there can be at most one pair of vertices in $\mathcal{L}$ and they can be linked through the vertex v. Assume that the vertices of X are labeled such that $\mathcal{L} = \{\{s_1, t_1\}, \{s_2, t_2\}\}$. Let H be the subgraph induced on $V(G) - \{v\} - X$. Then $|V(H)| \leq 5$. If H is not connected, then it has a component of order at most two. Consequently, there exists a vertex $x \in V(H)$ with at least four neighbors in X, and there exists an index $i = 1$ or 2 such that x is adjacent to both s_i and t_i. Then s_i and t_i can be linked through the vertex x and the other pair of vertices in $\mathcal{L}$ can be linked through the vertex v.

Thus we have shown that H must be a connected subgraph. If s_1 were adjacent to t_1, we could link s_2 and t_2 through the vertex v to show that the linkage problem is feasible. Otherwise, s_1 and t_1 are not neighbors, and consequently they each have at least two neighbors in H. Since H is connected, we can link s_1 and t_1 in the subgraph H and still link s_2 and t_2 through the vertex v. Thus the linkage problem $\mathcal{L}$ is feasible, completing the proof. $\square$

Lemma 5.4 *Let G and $X \subseteq V(G)$ with $|X| = 6$ such that $\delta(G) \geq 5$ and $|V(G)| \leq 9$. Let $\mathcal{L} = \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$ be a linkage problem on X. If $(G, X, \mathcal{L})$ is not quasi-firm, then the following hold:*

1. *For any vertices x_i and x_j in X, there exists a path linking x_i and x_j with no internal vertex in X,*
2. *for any linkage problem $\mathcal{L}'$ on X distinct from $\mathcal{L}$, the triple $(G, X, \mathcal{L}')$ is quasi-firm, and*
3. *for any index $i \in \{1, \ldots, 6\}$ and any vertex $y \in V(G) - X$, if we consider the linkage problem $\mathcal{L}' = \{\{y, x_{i+3}\}, \{x_{i+1}, x_{i+4}\}, \{x_{i+2}, x_{i+5}\}\}$ on $(X - \{x_i\}) \cup \{y\}$ where all index addition is mod 6, then $(G, (X - \{x_i\}) \cup \{y\}, \mathcal{L}')$ is quasi-firm.*

Proof: We prove the lemma by a series of intermediate claims. First, we prove several general observations about the structure of G before we analyze the cases arising from the possible sizes of G. Let H be the induced subgraph on $V(G) - X$, and let the vertices of H be labeled $h_1, \ldots, h_i$ where $i \leq 3$.

Claim 5.5 *H is a connected subgraph.*

Proof: Assume H is not connected. Because $|V(H)| \leq 3$, one component of H must then consist of an isolated vertex, call it h_1. Then h_1 has at least five neighbors in X, and consequently, there exist distinct indices i and j such that x_i, x_{i+3}, x_j and x_{j+3} all are adjacent to h_1. Also, there exists some h_2 distinct from h_1 that has at most one neighbor in H. Consequently h_2 has at least four neighbors in X, and so there exists an index k such that x_k and x_{k+3} are both adjacent to h_2. The index k must be distinct from i or j, so without loss of generality assume $k \neq i$. Then the paths $x_i h_1 x_{i+3}$ and $x_k h_2 x_{k+3}$ contradict our assumption that $(G, X, \mathcal{L})$ is not quasi-firm. $\square$

Conclusion 1 follows easily now.

Claim 5.6 *For any x_i and x_j in X, there exists a path linking x_i and x_j with no internal vertex in X.*

Proof: We may assume that x_i is not adjacent x_j. Then x_i and x_j each must have some neighbor in H. By Claim 5.5, H is connected so the desired path exists. □

Claim 5.7 *For every $i = 1, 2, 3$, the vertices x_i and x_{i+3} are not adjacent.*

Proof: Assume, without loss of generality, that x_1 is adjacent x_4. Using Claim 5.6, there exists a path linking x_2 and x_5, contradicting the assumption that $(G, X, \mathcal{L})$ is not quasi-firm. □

We have seen that H is connected, but in fact we can show something stronger. We now prove the following claim.

Claim 5.8 *H is a complete subgraph.*

Proof: Assume that H is not a complete subgraph. By Claim 5.5, we may assume that H is connected, forcing H to be a path on three vertices. Without loss of generality, assume that h_1 and h_3 are the endpoints of the path. Then h_1 and h_3 have four neighbors in X, and consequently there exists an index i such that h_1 is adjacent x_i and x_{i+3}. Similarly, there exists an index j such that h_3 is adjacent to x_j and x_{j+3}. We may assume that $i = j$, since otherwise the paths $x_i h_1 x_{i+3}$ and $x_j h_2 x_{j+3}$ contradict our assumption that $(G, X, \mathcal{L})$ is not quasi-firm. Without loss of generality, we assume $i = 1$ and x_1 and x_4 are both adjacent to h_1 and h_3. We know that h_2 must have at least three neighbors in X, so h_2 has some neighbor that is neither x_1 nor x_4. Without loss of generality, assume that x_2 is adjacent to h_2. The vertex x_5 has some neighbor in $V(H)$. If x_5 is adjacent to h_2, we get the linkage $x_1 h_1 x_4$ and $x_2 h_2 x_5$. But otherwise, x_5 is adjacent one of h_1 and h_3. The cases are symmetric, so assume x_5 is adjacent h_1. Then we get the linkage $x_1 h_3 x_4$ and $x_2 h_2 h_1 x_5$. Every case contradicts the assumption that $(G, X, \mathcal{L})$ is not quasi-firm, proving the claim. □

It will be convenient to refer to pairs of vertices we have shown to not be adjacent.

Definition A set $a = \{x, y\}$ of two distinct vertices x and y is an *anti-edge* if x is not adjacent to y.

To avoid confusion with edges, we will denote an anti-edge containing x and y by (x, y). An *anti-matching of size k* is a set of k disjoint anti-edges. A *perfect anti-matching* in a graph H is an anti-matching of size $|V(H)|/2$.

Claim 5.9 *$G[X]$ does not contain two distinct perfect anti-matchings.*

Proof: We know by Claim 5.7 that the pairs $x_1 x_4$, $x_2 x_5$, and $x_3 x_6$ form a perfect anti-matching. If another distinct perfect anti-matching on X existed, then there would exist two distinct indices i and j such that x_i, x_{i+3}, x_j and x_{j+3} all have at most three neighbors in X. Thus they each have at least two neighbors in H. Then x_i and x_{i+3} have a common neighbor in H, say h_1. By Claim 5.8, the subgraph $H - h_1$ is connected.

Since x_j and x_{j+3} each have a neighbor in $H - h_1$, we get the linkage consisting of $x_i h_1 x_{i+3}$ and a path from x_j to x_{j+3} with interior in $H - h_1$, a contradiction. $\square$

In other words, if $G[X]$ does not contain a unique perfect anti-matching, then $(G, X, \mathcal{L})$ is quasi-firm. The second conclusion of the lemma now follows easily.

Claim 5.10 *For any linkage problem $\mathcal{L}'$ on X distinct from $\mathcal{L}$, the triple $(G, X, \mathcal{L}')$ is quasi-firm.*

Proof Assume that $(G, X, \mathcal{L}')$ is not quasi-firm. Then Claim 5.7 holds for the triple $(G, X, \mathcal{L}')$. However, then both $\mathcal{L}$ and $\mathcal{L}'$ induce distinct perfect anti-matchings in X, contrary to Claim 5.9. $\square$

This proves Conclusion 2 of the lemma. We also can now prove the third point in the lemma.

Claim 5.11 *For any index $i \in \{1, \ldots, 6\}$ and any vertex $y \in V(G) - X$, if we consider the linkage problem $\mathcal{L}' = \{\{y, x_{i+3}\}, \{x_{i+1}, x_{i+4}\}, \{x_{i+2}, x_{i+5}\}\}$ on $(X - \{x_i\}) \cup \{y\}$ where all index addition is mod 6, then $(G, (X - \{x_i\}) \cup \{y\}, \mathcal{L}')$ is quasi-firm.*

Proof Assume the claim is false and that $(G, (X - \{x_i\}) \cup \{y\}, \mathcal{L}')$ is not quasi-firm. Without loss of generality, assume $i = 1$ and $\mathcal{L}' = \{\{y, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$. By the previous claims, we know that H is connected, and that x_1 is not adjacent x_4, which forces x_4 to have at least one neighbor in H. If x_2 and x_5 or x_3 and x_6 had x_1 as a common neighbor, say x_2 and x_5, we would get the path $x_2 x_1 x_5$ and we can connect y_1 to x_4 using H, contradicting the fact that $(G, (X - \{x_1\}) \cup \{y\}, \mathcal{L}')$ is not quasi-firm. Hence x_1 is adjacent to at most one vertex of x_2 and x_5 and at most one vertex of x_3 and x_6. Without loss of generality assume x_1 is not adjacent x_2 and x_3. By the minimum degree condition of G, it follows then that x_1 has three neighbors in H and that x_1 is adjacent to x_5 and x_6. From Claim 5.7 applied to the triple $(G, (X - \{x_1\}) \cup \{y\}, \mathcal{L}')$, we deduce that x_4 is not adjacent to y. It follows that x_4 must have a neighbor h_1 in H different from y. Let h_2 be the other vertex of H not equal to h_1 or y. Note that y is adjacent h_1 and h_2 by Claim 5.8.

If the vertex x_2 is adjacent to h_2, then the linkage $x_2 h_2 x_1 x_5$ and $y h_1 x_4$ contradicts the fact that $(G, (X - \{x_1\}) \cup \{y\}, \mathcal{L}')$ is not quasi firm. Thus x_2 is not adjacent to h_2 and by the minimum degree condition, x_2 is adjacent to y. Similarly, h_2 is not adjacent to x_3 and x_3 is adjacent to y. The vertex h_2 must be adjacent to one of x_5 and x_6, again by the minimum degree condition of G. By symmetry, assume h_2 is adjacent x_5. We get the linkage $x_2 y h_2 x_5$ and $x_1 h_1 x_4$, contradicting the fact that the triple $(G, X, \mathcal{L})$ is not quasi-firm. This final contradiction proves the claim. $\square$

This completes the proof of the lemma. $\square$

A significant difficulty in the proof of Theorem 5.1 is what to do with separations (A, B) of order six where $X \subseteq A$ and $\rho(B - A) > 5|B - A|$ and $\rho(B - A) < 5|B - A| + 4$. If $\rho(B - A) > 5|B - A|$, then $B - A$

contains too many edges to simply disregard, however, with $\rho(B-A) < 5|B-A|+4$, $G[B]$ does not contain enough edges to provide a rigid separation. However, in the application in the proof, the separation (A, B) will be such that $(G[B], A \cap B)$ is 2-linked, and we will be able to proceed.

Moreover, these unpleasant separations need not be unique. We will have to examine the case when the graph can be decomposed into a large number of non-crossing separations. We explicitly define a decomposition thus:

Definition Let $X \subseteq V(G)$ with $|X| = 6$ and let $k \geq 1$. A sequence $(A, B_1, \ldots, B_k)$ of subsets of $V(G)$ is a *star decomposition of* (G, X) if the following conditions hold:

1. $X \subseteq A$,
2. for all distinct indices $i, j \in \{1, \ldots, k\}$, $B_i \cap B_j \subseteq A$,
3. for all $i \in \{1, \ldots, k\}$, $(\bigcup_{j \neq i} B_j \cup A, B_i)$ is a separation of order exactly six, and
4. for all $i \in \{1, \ldots, k\}$, $(G[B_i], A \cap B_i)$ is 2-linked.

The separations $(\bigcup_{j \neq i} B_j \cup A, B_i)$ are called the separations *determined* by the star decomposition $(A, B_1, \ldots, B_k)$.

As an easy observation about star decompositions, we give the following lemma.

Lemma 5.12 *Let $(G, X, \mathcal{L})$ be a 3-minimal triple, and let $(A, B_1, \ldots, B_k)$ be a star decomposition of (G, X). For all $i = 1, \ldots, k$, there does not exist a separation (C, D) of G of order at most five with $X \subseteq C$ and $B_i \subseteq D$.*

Proof: Assume such a separation (C, D) existed for some index i. Assume we pick such a separation of minimal order. Then there exist disjoint paths from $C \cap D$ to $B_i \cap A$ in $G[D]$, and any linkage problem on $C \cap D$ extends to a linkage problem on $B_i \cap A$. Moreover, since $|C \cap D| \leq 5$, the induced linkage problem has at most two pairs of vertices. Consequently, the induced linkage problem will be feasible in $G[B_i]$, implying that (C, D) is a rigid separation of (G, X). This contradicts Lemma 5.2. $\square$

Given a pair (G, X) and a star decomposition $(A, B_1, \ldots, B_k)$, let $e = uv$ be a fixed edge of G not contained in X. Then the star decomposition $(A, B_1, \ldots, B_k)$ *induces* the star decomposition $(A^*, B_1^*, \ldots, B_k^*)$ in G/e where v_e, the vertex of G/e corresponding to the contracted edge e, lies in B_i^* or A^* if and only if either u or v are elements of B_i or A, respectively.

Lemma 5.13 *Let $(G, X, \mathcal{L})$ be a 3-minimal triple. Let $e = uv$ be a fixed edge in G not contained in X. Let G have a star decomposition $(A, B_1, \ldots, B_k)$ with the added constraint that $e \subseteq B_i \cap A$ for all $i = 1, \ldots, k$. Let $(A^*, B_1^*, \ldots, B_k^*)$ be the induced star decomposition in G/e. Then $\bigcup_i (B_i^* \cap A^*)$ has at least $3k$ anti-edges.*

The proof of Lemma 5.13 is somewhat involved and technical. We postpone the proof until Section 6 and proceed with the proof of Theorem 5.1.

Proof of Theorem 5.1, assuming Lemma 5.13. Assume the theorem is false. We let $(G, X, \mathcal{L})$ be a 3-minimal triple. First, we make the following observation.

Claim 5.14 *The pairs (x_i, x_{i+3}) are anti-edges for all $i = 1, 2, 3$. Moreover, these are the only anti-edges in $G[X]$.*

Proof: If there exists some index, say $i = 1$, such that x_1 is adjacent to x_4, then by Lemma 3.3 (*iii*), the linkage problem $\mathcal{L}$ is feasible in G, a contradiction. Moreover, since adding any edge to $G[X]$ not linking a pair of vertices of $\mathcal{L}$ does not affect the feasibility of $\mathcal{L}$, we see by (E) in the definition of 3-minimality that every anti-edge of $G[X]$ is of the form (x_i, x_{i+3}) for some index i. $\square$

Claim 5.15 *Every edge e, where $e \not\subseteq X$, the edge e is contained in at least five triangles.*

Proof Assume $e = uv$ is such an edge but that the endpoints of e do not have five common neighbors. Contract the edge e. If the pair $(G/e, X)$ is $(5, 4)$-massed, then by minimality, $\mathcal{L}$ is feasible in G/e. The paths solving $\mathcal{L}$ extend to paths in G, contradicting the fact that $\mathcal{L}$ is not feasible in G. It follows that $(G/e, X)$ fails to satisfy (M1) or (M2). We claim it fails the latter. To prove this claim, suppose for a contradiction that $(G/e, X)$ satisfies (M2); then it does not satisfy (M1). Thus $\rho_G(V(G) - X) - \rho_{G/e}(V(G/e) - X) \geq 6$. If e does not have an end in X, the number $\rho_{G/e}(V(G/e) - X)$ decreases by the number of common neighbors of u and v plus one. By our assumptions on e, then, either u or v must be a vertex of X. In this case, $\rho_{G/e}(V(G/e) - X)$ decreases by the number of triangles containing e plus the number of neighbors of v in $X - \{u\}$ not adjacent to u. By (E) in the definition of 3-minimal, the vertex u has at most one non-neighbor in X. It follows that $\rho_{G/e}(V(G/e) - X) \geq 5|V(G/e) - X| + 3$ and if equality holds, there exists an index i such that x_i and x_{i+3} in X are adjacent in G/e. Since the pair $(G/e, X)$ satisfies (M2), either $(G/e, X)$ is $(5, 4)$-massed, or $(G/e, X)$ is $(5, 3)$-massed and x_i is adjacent to x_{i+3}. The linkage problem $\mathcal{L}$ is feasible in G/e by minimality in the first case; $\mathcal{L}$ is feasible by Lemma 3.3 (*iii*) in the second case. Either is a contradiction. This proves the claim, and we conclude that $(G/e, X)$ fails to satisfy (M2).

Then G/e has a separation (A^*, B^*) of order at most five with $\rho(B^* - A^*) \geq 5|B^* - A^*| + 1$. We will use the separation (A^*, B^*) to construct a star decomposition of (G, X). Note that (A^*, B^*) is a rigid separation of $(G/e, X)$ by Lemma 3.3 (*i*). This separation induces a separation (A, B) in G in the following manner. Let $v_e \in V(G/e)$ be the vertex corresponding to the contracted edge, and then $A = (A^* \cup \{u, v\}) - \{v_e\}$ if $v_e \in A^*$ and $A = A^*$ otherwise. Similarly define B. First consider the case when $e \not\subseteq A \cap B$. If the edge $e \subseteq A$, then (A, B) is a separation in G violating (M2). Now assume $e \subseteq B$. Then (A, B) is a rigid separation of (G, X), since any paths linking $A^* \cap B^*$ in G/e also exist in G. This is a contradiction to Lemma 5.2. We conclude that $e \subseteq A \cap B$. Note that in G, $\rho(B - A) \geq 5|B - A| + 1$. Consequently $|A \cap B| = 6$. By Lemma 3.3, we know that (A, B) is a 2-linked separation of (G, X) with $e \subseteq A \cap B$.

A 2-linked separation (A, B) of (G, X) of order six with $e \subseteq A \cap B$ is *maximal* if there does not exist a separation (A', B') of (G, X) of order six with $e \subseteq A' \cap B'$ and $X \subseteq A' \subsetneq A$. Since (G, X) has at least one 2-linked separation with e contained in the intersection, it must have at least one maximal 2-linked separation of order six. Let $(A, B_1, \ldots, B_k)$ be a star decomposition of (G, X), where each separation determined by the star decomposition is maximal and $e \subseteq A_i \cap B_i$ for all i. Assume we have chosen the decomposition such that k is maximum. Let $A^*, B_1^*, \ldots, B_k^*$ be the sets of vertices induced in G/e by $(A, B_1, \ldots, B_k)$, and again let $S_i^* = B_i^* \cap A^*$. We know that $\rho(B_i - S_i) \leq 5|B_i - S_i| + 3$, lest by minimality we find a rigid separation of order six. Thus in G/e, $\rho(B_i^* - S_i^*) \leq 5|B_i^* - S_i^*| + 3$. By Lemma 5.13, we see that in G/e that there are a total of $3k$ anti-edges contained in $\bigcup S_i^*$. Moreover, each $(\bigcup_{j \neq i} B_j^* \cup A^*, B_i^*)$ is a rigid separation of G/e.

Consider G^* defined by taking G/e and deleting all vertices in $B_i^* - S_i^*$ for every i and adding edges to all non-adjacent pairs in any S_i^*. First observe that any linkage solving $\mathcal{L}$ in G^* would extend to a linkage in G solving $\mathcal{L}$. That is because if we picked such a linkage to minimize the number of vertices used, each path would use at most one edge in any S_i^* since $G^*[S_i^*]$ is complete. Moreover, since $|S_i^*| \leq 5$, at most two paths in our solution use edges contained in S_i^*. Then looking at the linkage solving $\mathcal{L}$ in G, we are missing at most two edges in S_i for any index i. Because the determined separations of a star decomposition are 2-linked, we can extend the solution of $\mathcal{L}$ in G^* to a solution in G, contradicting the definition of 3-minimality.

We now prove that the pair (G^*, X) satisfies (M2). Assume we have a separation (C, D) in (G^*, X) violating (M2). Pick such a separation to minimize $|C|$. If $v_e \in C - D$, then every $S_i^* \subseteq C$, and as a consequence, $((C - \{v_e\}) \cup \{u, v\} \cup (\bigcup_i B_i), D)$ is a separation in G violating (M2). If $v_e \subseteq D - C$, then $(C, (D - \{v_e\}) \cup \{u, v\} \cup (\bigcup_i B_i))$ is a rigid separation in G because disjoint paths linking $C \cap D$ in G^* extend to disjoint paths in G as in the previous paragraph. Thus we may assume $v_e \in C \cap D$. Then no S_i^* is a subset of D, lest we violate the maximality of the separation $(A \cup (\bigcup_{j \neq i} B_j), B_i)$ or Lemma 5.12. Also, we know that $|C \cap D| = 5$, lest (G, X) have a separation violating (M2). It follows that $((C - \{v_e\}) \cup \{u, v\}, B_1, \ldots, B_k, (D - \{v_e\}) \cup \{u, v\})$ is a star decomposition of G violating our choice to make k maximum. This completes the proof that (G^*, X) satisfies (M2).

We now count $\rho_{G^*}(V(G^*) - X)$ and show that $\mathcal{L}$ must be feasible in G^*, contradicting our earlier observation that a linkage solving $\mathcal{L}$ in G^* extends to a linkage solving $\mathcal{L}$ in G. In our initial observations for this claim, we saw that $\rho_{G/e}(V(G/e) - X) \geq 5|V(G/e) - X| + 3$ with equality holding if and only if there exists an index i such that x_i is adjacent to x_{i+3} in G/e. When we construct G^* and we delete the vertices of $B_i^* - A^*$, we lose at most $5|B_i^* - A^*| + 3$ edges for $i = 1, \ldots, k$. By Lemma 5.13, $|E(G^*)| \geq |E(G/e[A^*])| + 3k$, which implies that $\rho_{G^*}(V(G^*) - X) - \rho_{G/e}(A^* - X)$ is at least $3k$ minus the number of edges added to G^* that have both ends in X. We conclude that $\rho_{G^*}(V(G^* - X) \geq 5|V(G^*) - X| + 4 - t$ where t is the number of indices i such that x_i is adjacent x_{i+3} in G^*. By the 3-minimality of $(G, X, \mathcal{L})$ if $t = 0$, or by Lemma 3.3 if $t \geq 1$, it follows that $\mathcal{L}$ is feasible in G^*, a contradiction. This completes the proof of the claim. $\square$

Claim 5.16 $\rho(V(G) - X) = 5|V(G) - X| + 4.$

Proof Consider an edge $e = uv$ such that $e \not\subseteq X$. If $(G - e, X)$ is $(5,4)$-massed, then by the definition of 3-minimality, there exist disjoint paths in $G - e$ solving the linkage problem $\mathcal{L}$. Those paths would exist in G as well, a contradiction. We conclude that $G - e$ violates (M1) or (M2).

Let (A, B) be a separation of $(G - e, X)$ violating (M2). Then without loss of generality, we may assume $u \in A - B$ and $v \in B - A$, lest (G, X) have a separation violating (M2). By Claim 5.15, we know u and v have at least five common neighbors. These neighbors must be in A because $u \in A - B$, and these neighbors must also be in B because $v \in B - A$. It follows that u and v are adjacent to every vertex of $A \cap B$, and $|A \cap B| = 5$. By considering the separation $(A, B \cup \{u\})$ in G, we see that $\rho((B \cup \{u\})) - A) \geq 5|(B \cup \{u\}) - A| + 2$, so $(G[B \cup \{u\}], (A \cap B) \cup \{u\})$ is 2-linked by Lemma 3.3 (iii). In fact, this is actually a rigid separation because u is adjacent every other vertex in $A \cap B$, so given any linkage problem on $(A \cap B) \cup \{u\}$, we can link u to its paired vertex with an edge and link the remaining two pairs of vertices with paths in $G[B]$. This contradicts Lemma 5.2. We conclude that $(G - e, X)$ violates (M1), implying the claim.□

Claim 5.17 *There exists a vertex $v \in V(G) - X$ such that $6 \leq \deg(v) \leq 9$, and in fact if no such vertex of degree at most seven exists, then there exist at least two vertices in $V(G) - X$ with degree either eight or nine.*

Proof The previous claim states that $\rho(V(G) - X) = 5|V(G) - X| + 4$. Observe that every vertex in X must have at least two neighbors in $V(G) - X$. If $x_i \in X$ had no neighbors in $V(G) - X$, then $(X, V(G) - \{x_i\})$ is a separation of order five violating (M2). If x_i had only one neighbor in $V(G) - X$, say the vertex y, then the edge $x_i y$ must be in five triangles. But x_i has no other neighbor in $V(G) - X$, so x_i and y must have five common neighbors in X, and consequently, x_i is adjacent to x_{i+3}, contrary to Claim 5.14. Hence, every vertex $x_i \in X$ has at least two neighbors in $V(G) - X$.

Define $f(x)$ to be the number of neighbors that $x \in X$ has in $V(G) - X$. Then

$$2\rho(V(G) - X) = \sum_{v \in V(G) - X} \deg(v) + \sum_{x \in X} f(x).$$

Suppose that every vertex of $V(G) - X$ has degree in G at least eight, and let k be the number of vertices of $V(G) - X$ of degree at most nine. Then

$$\sum_{v \in V(G) - X} \deg(v) + \sum_{x \in X} f(x) \geq 10(|V(G) - X| - k) + 8k + 2|X|.$$

Claim 5.16 implies that the left-hand side is equal to $2(5|V(G) - X| + 4)$, and hence $k \geq 2$, as desired. □

Now we will see that either the linkage problem $\mathcal{L}$ is feasible in G contradicting the fact that $(G, X, \mathcal{L})$ is a 3-minimal pair, or we find a separation violating Lemma 5.2.

Claim 5.18 *There do not exist two vertices in $V(G) - X$ each adjacent to every vertex of X.*

Proof Assume the claim is false, and let u and v be two such vertices. Then consider connected components $A_1, \ldots, A_t$ of $G - (X \cup \{u, v\})$. Then $\rho_G(V(G) - X) = \rho_{G[X \cup \{v,u\}]}(\{v, u\}) + \sum_{i=1}^{k} \rho(A_i)$. Since $\rho_{G[X \cup \{v,u\}]}(\{v, u\}) \leq 13 = 5(2) + 3$, we see that $\rho_G(A_i) \geq 5|A_i| + 1$ for some index i. Then A_i must have at least six neighbors in $X \cup \{v, u\}$, implying that A_i must have four neighbors among X. Thus there exists an index j such that A_i has a neighbor of x_j and x_{j+3}. Then the linkage problem $\mathcal{L}$ is feasible since we can link one pair with A_i and the other two pairs with u and v, a contradiction. □

Now we examine the neighborhood of a vertex of small degree in $V(G) - X$. Let $v \in V(G) - X$ be such a vertex of degree equal to the minimum of 6 or 7, if possible, and otherwise, pick v to be a vertex of degree at most 9, and if possible, pick it such that it is not adjacent every vertex of X. As we saw above, such a vertex exists. Let N be the subgraph induced on $N(v) \cup \{v\}$.

Claim 5.19 *There exist six disjoint paths linking X to $N(v)$.*

Proof Assume the paths do not exist. Then there exists a separation (A, B) of order at most five with $X \subseteq A$ and $N(v) \subseteq B$. Pick such a separation of minimal order. Then there exist a linkage $\mathcal{Q}$ with $|A \cap B|$ components from $A \cap B$ to $N(v)$. We may assume that no component of $\mathcal{Q}$ uses the vertex v. Let X' be the termini of the components of $\mathcal{Q}$ in $N(v)$. By Lemma 5.3 and Claim 5.15, the pair (N, X') is linked. Consequently, the separation (A, B) is rigid contrary to Lemma 5.2. This proves the claim. □

Let $\mathcal{P}$ be a linkage from X to $N(v)$. Label the components of $\mathcal{P}$ $P_1, \ldots, P_6$ and label the termini of $\mathcal{P}$ such that the endpoints of P_i are x_i and x'_i. Let $X' := \{x'_1, x'_2, \ldots, x'_6\}$ and let $\mathcal{L}'$ be the linkage problem on X' induced by $\mathcal{L}$ and $\mathcal{P}$.

If $(G[N(v)], X', \mathcal{L}')$ were quasi-firm, then $\mathcal{L}$ would be feasible, a contradiction. Thus the conclusions of Lemma 5.4 hold for the triple $(G[N(v)], X', \mathcal{L}')$. If there exists a path from $V(\mathcal{P}) - X'$ to $N(v) - X'$, then we can reroute some path P_i to arrive in $N(v)$ in some vertex y not contained in X'. As a result, we have a linkage from X to $N(v)$ with the set of termini being $(X' - \{x_i\}) \cup \{y\}$ such that the linkage problem $\mathcal{L}'$ induced by $\mathcal{L}$ is $\{\{y, x_{i+3}\}, \{x_{i+1}, x_{i+4}\}, \{x_{i+2}, x_{i+5}\}\}$ with all index addition mod six. Then by Lemma 5.4, $(G[N(v)], (X' - \{x_i\}) \cup \{y\}, \mathcal{L}')$ is quasi-firm, implying that $\mathcal{L}$ is feasible in G, a contradiction.

We conclude that there exists a separation (A, B) with $X \subseteq A$, $V(N) \subseteq B$, and $A \cap B = X'$. Assume that the separation (A, B) is non-trivial. Then we see that $\rho(B - A) \leq 5|B - A| + 3$, lest (A, B) be a rigid separation by the minimality of $(G, X, \mathcal{L})$. Consequently, $\rho_{G[A]}(A - X) \geq 5|A - X| + 1$. We apply Lemma 4.4 to the subgraph $G[A]$ and the linkage $\mathcal{P}$ from X to $A \cap B$. If (C1) holds, then we can link the remaining two pairs of vertices in $\mathcal{L}'$ by property 1. in Lemma 5.4 and using the vertex v adjacent all of X'. If (C2) holds in the application of Lemma 4.4, then there exists a linkage $\mathcal{P}'$ from X to $A \cap B$ inducing a distinct linkage problem on X'. But then by property 2. Lemma 5.4, this new linkage problem is feasible in $G[B]$. Either case gives a contradiction to the fact that $\mathcal{L}$ is not feasible in G.

If the separation (A, B) in the previous paragraph were in fact trivial, then the vertex v is adjacent to every vertex of X. If $|N(v)| = 6$, then $G[X]$ is a complete subgraph, a contradiction. If $|N(v)| = 7$, then since there does not exist an index i with x_i adjacent to x_{i+3}, Claim 5.15 implies that $N(v)$ consists of X and a single vertex adjacent to all of X. This contradicts Claim 5.18. Thus we see $\deg(v) \geq 8$. But then there is at least one more such vertex u of degree at most nine by Claim 5.17. By the choice of v, the vertex u is also adjacent to every vertex of X, again contradicting Claim 5.18. This final contradiction completes the proof of Theorem 5.1 demonstrating that no 3-minimal triple exists. $\square$

6 Proof of Lemma 5.13

Given the star decompositions in the statement, let $S_i := A \cap B_i$ and $S_i^* = B_i^* \cap A^*$. The proof of the lemma will follow from two main arguments. First, since every B_i determines a 2-linked separation, we will see that every S_i must contain several anti-edges. In fact, we will see that even upon contracting the edge e, S_i will contain three anti-edges. We will show that these anti-edges can be chosen to be pairwise distinct for different values of the index i.

Claim 6.1 *For all $i = 1, 2, \ldots, k$ we have $\rho(B_i - S_i) \leq 5|B_i - S_i| + 3$.*

Proof: Otherwise the separation $(\bigcup_{j \neq i} B_j \cup A, B_i)$ is rigid by the 3-minimality of $(G, X, \mathcal{L})$, contrary to Lemma 5.2. $\square$

Claim 6.2 *For every value of $i \in \{1, 2, \ldots, k\}$, $G[S_i]$ has two distinct perfect anti-matchings. For any anti-edge (x, y) of either of the two anti-matchings, there exists a linkage $\mathcal{P}$ from X to S_i with six components and an index j such that if we label P_k the component of $\mathcal{P}$ containing x_k, then the termini of P_j and P_{j+3} are x and y.*

Proof: By Lemma 5.12, there exist six disjoint paths from X to S_i. Given a linkage from X to S_i, the linkage problem $\mathcal{L}$ induces a linkage problem $\mathcal{L}'$ on S_i. Each pair of $\mathcal{L}'$ must be an anti-edge, lest we link the two remaining pairs in $G[B_i]$ and contradict the fact that $\mathcal{L}$ is not feasible.

Given that $\rho(G-X) \geq 5|G-S|+4$, Claim 6.1 implies that $\rho\left(\left(\bigcup_{j \neq i} B_j \cup A\right) - X\right) \geq 5|\left(\bigcup_{j \neq i} B_j \cup A\right) - X|+1$. By Lemma 4.4, one of (C1) or (C2) must hold. If (C1) holds, we can link one pair of $\mathcal{L}$ in $G[\bigcup_{j \neq i} B_j \cup A]$ and link the two remaining pairs in $G[B_i]$, making $\mathcal{L}$ feasible, a contradiction. Thus (C2) holds. Through this new linkage from X to S_i, $\mathcal{L}$ induces a linkage problem $\mathcal{L}''$ distinct from $\mathcal{L}'$. As in the previous paragraph, the pairs in $\mathcal{L}''$ form an perfect anti-matching. Thus $G[S_i]$ contains two distinct perfect anti-matchings, and the claim follows. $\square$

We now examine in more depth the properties of $G[S_i]$.

Claim 6.3 *Fix i and let $x, y \in S_i$. Then $G[S_i]$ has at least two anti-edges not incident with x or y, and moreover, if there are exactly two such edges, then they have a common end point.*

Proof: By Claim 6.2 the complement of the graph $G[S_i]$ has a subgraph isomorphic to C_6 or $C_4 \cup K_2$. Thus $G[S_i]$ must contain at least one anti-edge not incident with x or y. We may assume that $G[S_i]$ has at least three anti-edges incident with x or y, for otherwise the conclusion holds.

Assume for every vertex v in $G[S_i]$, there is at most one anti-edge incident with v that does not have x or y as the other endpoint. Let the graph G' obtained from G by deleting the vertices $B_i - S_i$ and for every $z \in S_i - \{x, y\}$, adding the edge xz and yz if it does not already exist, and adding the edge xy if it does not already exist. By Claim 6.1 and the fact that S_i had at least three anti-edges incident with x or y, we know that $\rho_{G'}(V(G') - X) \geq 5|V(G') - X| + 4$. Now if (G', X) had no separation violating (M2), then by the 3-minimality of $(G, X, \mathcal{L})$, the pair (G', X) is linked. Let P_1, P_2, and P_3 be paths solving the linkage problem $\mathcal{L}$. At most two of these paths use the vertices x and y, so we may assume P_3 uses only edges present in G. If either the paths P_1 and P_2 contain vertices of S_i, then they have first and last vertices in S_i. Label the vertices w_1, w_2 for P_1, and z_1, z_2 for P_2. In $G[B]$, there exist paths Q_1 and Q_2 with ends w_1, w_2 and z_1, z_2 respectively, with the property that $Q_i \cap V(G') \subseteq S_i$. Then $x_1 P_1 w_1 Q_1 w_2 P_1 x_4$, $x_2 P_2 z_1 Q_2 z_2 P_2 x_5$ and $x_3 P_3 x_6$ is a linkage in G solving $\mathcal{L}$. This contradiction implies that (G', X) has a separation violating (M2).

Let (A', B') be a separation in (G', X) violating (M2). Then if $S_i \subseteq A'$, then $(A' \cup B_i, B')$ is a separation of (G, X) violating (M2). If $S_i \subseteq B'$, then we would have a separation of order at most five separating X from S_i, contradicting Lemma 5.12. It follows that there exist some vertices w_1 and w_2 in S_i such that $w_1 \in A' - B'$ and $w_2 \in B' - A'$. Then w_1 is not adjacent w_2, and by our assumptions on $G[S_i]$, we know w_1 and w_2 are each adjacent (in G') to every other vertex in S_i. If the other vertices of S_i are x, y, z_1, z_2, then $x, y, z_1, z_2 \in A' \cap B'$. If $A' \cap B' = \{x, y, z_1, z_2\}$, then $(A' \cup \{w_2\} \cup B_i, B')$ is a separation in G of order five separating X from B_i, a contradiction again to Lemma 5.12. We conclude $A' \cap B'$ contains exactly one other vertex not yet defined. Call it a. In the graph G', there exist six disjoint paths from X to $\{x, y, z_1, z_2, a, w_1\}$, lest G have a separation of order at most five separating X from S_i. Label the six paths $P_1, \dots, P_6$ and let the ends of P_j be $x_j \in X$ and $x'_j \in \{x, y, z_1, z_2, a, w_1\}$. Note P_j may be a trivial path consisting of just one vertex, in which case x_j and x'_j are not distinct.

The linkage problem $\mathcal{L}$ induces the linkage problem $\mathcal{L}' = \{\{x'_1, x'_4\}, \{x'_2, x'_5\}, \{x'_3, x'_6\}\}$ on $\{x, y, z_1, z_2, w_1, a\}$. We now show that the linkage problem $\mathcal{L}'$ is feasible in $G[B' \cup B_i]$, contradicting the fact that $\mathcal{L}$ is not feasible in G. Some pair of vertices in the linkage problem $\mathcal{L}'$ lies in $\{x, y, z_1, z_2\}$. Without loss of generality, say $x'_1, x'_4 \in \{x, y, z_1, z_2\}$. Then in G there exist paths Q_1, Q_2 with all internal vertices in B_i with the ends of Q_1 being w_1 and w_2 and the ends of Q_2 being x'_1 and x'_4. Now there are two cases to consider.

Case 1: w_2 is adjacent to every vertex in $A' \cap B'$. In this case $\rho_G(B' - A' - \{w_2\}) \geq 5|B' - A' - \{w_2\}| + 1$, and as a consequence, G restricted to $B' - A' - \{w_2\}$ has some connected component C with $\rho(V(C)) \geq$

$5|V(C)|+1$. This implies the vertices of $(A' \cap B') \cup \{w_2\}$ all have a neighbor in the component C, lest (G, X) have a separation violating (M2). We can link the path end w_1 with its paired vertex in $\mathcal{L}'$ via w_2 and the path Q_1, x_1' and x_4' via the path Q_2 and the remaining pair of vertices in $\mathcal{L}'$ via the connected component C. This would make $\mathcal{L}'$ feasible in $G[B_i \cup B' \cup \{w_1\}]$, a contradiction.

Case 2: w_2 has at least one non-neighbor in $A' \cap B'$. In this case, $\rho(B' - A' - \{w_2\}) \geq 5|B' - A' - \{w_2\}| + 2$. Then by Lemma 3.3, we know any two path problem on $\{x, y, w_2, z_1, z_2, a\}$ can be solved with disjoint paths with all internal vertices in $B' - A' - \{w_2\}$. We can link x_1' and x_4' in B_i with the path Q_1. By linking w_1 to w_2 with Q_2, we can link the remaining two pairs of vertices in $\mathcal{L}'$ in $B' - A' - \{w_2\}$ to show that the linkage problem $\mathcal{L}'$ is feasible.

This completes the proof of Claim 6.3 that $G[S_i]$ must have at least two anti-edges not incident with e, and if it has exactly two, then they must share a common endpoint. $\square$

Recall, the edge $e = uv$ lies in every S_i of our star decomposition.

Claim 6.4 *Fix i. If $G[S_i]$ has exactly two anti-edges not incident with e, then we can label the anti-edges a_1 and a_2 and label the underlying vertices $a_1 = (x, y)$, $a_2 = (y, z)$ such that*

1. *There exists a linkage $\mathcal{P}$ from X to S_i with six components and an index j such that if we label P_k the component of $\mathcal{P}$ containing x_k, then a_1 contains the two endpoints of P_j and P_{j+3} in S_i, and*
2. *the vertex z is a common non-neighbor of the ends of e.*

Proof: By Claim 6.2, the complement of $G[S_i]$ has a subgraph A isomorphic to C_6 or $C_4 \cup K_2$. If $G[S_i]$ has exactly two anti-edges not incident e, then there are three possible cases, up to isomorphism, for how the edge $e = uv$ intersects with A.

First, assume that A is isomorphic to $C_4 \cup K_2$. Let the vertices of S_i be labeled c_1, c_2, c_3, c_4 corresponding to the C_4 in order and k_1, k_2 corresponding to the K_2.

Case 1: $u = c_1, v = k_1$. In this case, one of the following pairs must be an anti-edge: $(k_1, c_2), (k_1, c_4)$, $(k_2, c_2), (k_2, c_4)$. Otherwise, when we consider the vertices c_3 and c_1, there would not exist at least two incident anti-edges with neither c_3 nor c_1 as an endpoint, contrary to Claim 6.3. Since $G[S_i]$ has exactly two anti-edges not incident with e, we may assume the pair (k_1, c_2) is an anti-edge. Then let $a_1 = (c_3, c_4)$ and $a_2 = (c_2, c_3)$. By Claim 6.2, there exists a linkage $\mathcal{P}$ from X to S_i such that if we label P_i the component of $\mathcal{P}$ containing x_i, then there exists an index j such that a_1 contains the two ends of P_j and P_{j+3} in S_i. As we have already seen that c_2 is a common non-neighbor of the ends of e, we have proven the claim.

Case 2: $u = c_2$, $v = c_4$. Again by Claim 6.3, there must be some other anti-edge not incident with e. Without loss of generality, it's the pair (k_1, c_1). Then if we let $a_1 = (k_1, k_2), a_2 = (k_1, c_1)$ we have the desired labeling of the anti-edges where now c_1 is the common non-neighbor of the ends of e. Again, by Claim 6.2, there exists a linkage from X to S_i where for some pair of the linkage problem $\mathcal{L}$, the corresponding paths terminate on the anti-edge a_1, as desired.

This completes the analysis when A is isomorphic to $C_4 \cup K_2$. Now we assume A is isomorphic to C_6. Let the vertices of S_i be labeled $c_1, c_2, \ldots, C_6$ in the order determined by A. There is only one possible choice, up to isomorphism, for the edge e such that there are only two anti-edges not incident e.

Case 3: $u = c_1$**,** $v = c_3$**.** By applying Claim 6.3 to the vertices c_2 and c_5, one of the following pairs must be an anti-edge: $(c_3, c_6), (c_6, c_4), (c_4, c_1)$. And since by assumption $G[S_i]$ has exactly two anti-edges not incident with e, we may assume that either (c_3, c_6) or (c_1, c_4) is an anti-edge. The two cases are symmetric, so we may assume (c_3, c_6) is an anti-edge, and then if we let $a_1 = (c_4, c_5)$ and $a_2 = (c_5, c_6)$, we have the desired properties. Again the existence of the required linkage follows from Claim 6.2.

This completes the proof of Claim 6.4. $\square$

Now we have a solid grip on what the subgraph $G[S_i]$ can look like; $G[S_i]$ must have at least two anti-edges not incident with e. Moreover, if there are exactly two such anti-edges, there is a common non-neighbor of the ends of e. Then clearly, upon contracting the edge e, $G/e[S_i^*]$ has at least three anti-edges. We will first show that if $|S_i \cap S_j| \geq 5$ for some $j \neq i$, then these anti-edges may be chosen so that they belong to no S_ℓ^* for $\ell = i$.

For notation, the next claims will be proven for S_1, S_2, and S_3. Since the labeling of the S_i's is arbitrary, we see that the results will hold for any distinct S_i, S_j, and S_k.

Claim 6.5 *Given S_1 and S_2 above, $|S_1 \cap S_2| \leq 4$. If $|S_1 \cap S_2| = 4$, then there exists a linkage $\mathcal{P}$ in G with six components from X to $S_1 \cup S_2$, where if we label P_i the component of $\mathcal{P}$ containing x_i, the following hold.*

1. *There exists an index i such that both P_i and P_{i+3} have their termini in $S_1 \cap S_2$.*
2. *No other component of $\mathcal{P}$ has its terminus in $S_1 \cap S_2$.*
3. *For indices $j \in \{1, \ldots, 6\}, j \neq i, i+3$, if the terminus of P_j lies in $S_1 - S_2$, then the terminus of P_{j+3} lies in $S_2 - S_1$, with all index addition mod 6.*
4. *At least one vertex of u and v is not the terminus of a component of $\mathcal{P}$.*
5. *$V(\mathcal{P}) \cap (S_1 \cup S_2)$ consists of the six termini of the components of $\mathcal{P}$.*

Proof Assume $|S_1 \cap S_2| \geq 4$. Clearly, $S_1 \neq S_2$, lest $(\bigcup_{j \neq 1,2} B_j \cup A, B_1 \cup B_2)$ form a rigid separation. It follows that $|S_1 \cap S_2|$ is at most five. There exists a linkage $\mathcal{P}$ from X to S_1. Let the component of $\mathcal{P}$ containing x_i be labeled P_i. Let the terminus of P_i in S_1 be labeled x_i'. The linkage problem $\mathcal{L}$ induces a linkage problem $\mathcal{L}'$ on S_1. If we can find paths solving $\mathcal{L}'$ that do not use any vertex of $\mathcal{P}$, except for their ends, then clearly we would contradict the fact that $\mathcal{L}$ is not feasible in G. Note that since $|S_1 \cap S_2| \geq 4$, there exists an index i such that P_i and P_{i+3} have their termini in $S_1 \cap S_2$. Without loss of generality, assume that $x'_1, x'_4 \in S_1 \cap S_2$.

If no path of $\mathcal{P}$ uses vertices of $B_2 - S_2$, then clearly we can link x'_1 and x'_4 with a path in $B_2 - S_2$ and link the two remaining pairs in $B_1 - S_1$. If at most one path, say P_l, uses vertices of $B_2 - S_2$, let x''_l be the first vertex of P_l in S_2. Then there are two cases. If $l = 1$ or 4, say $l = 1$, instead of following P_l to x'_1, instead find a path in B_2 from x''_1 to x'_4. Link the remaining two pairs of vertices in $\mathcal{L}'$ in B_1. Now assume $l \neq 1$ or 4. Let y be the final vertex of P_l in S_2. Then find paths in B_2 solving the linkage problem $\{\{x''_l, y\}, \{x'_1, x'_4\}\}$. Link the remaining two pairs of vertices in $\mathcal{L}'$ in B_1. Either case gives rise to a contradiction. We conclude that $|S_1 \cap S_2| = 4$, and that exactly two paths, say P_l and P_k, use vertices of $B_2 - S_2$. Again, let x''_k and x''_l be the first vertices in $S_2 - S_1$ of P_k and P_l, respectively. Then if $k = j + 3$, or $j = k + 3$, then we can link x_k and x_j with a path in B_2 and link the remaining two pairs of terminals in $\mathcal{L}'$ in B_1, a contradiction. Without loss of generality, we assume $k = 1$ and $l = 2$. The paths P_1, P_2 each use a vertex of $S_2 - S_1$, so it follows that x'_1 and x'_2 lie in $S_1 \cap S_2$.

Let $\mathcal{P}'$ be the linkage $(\mathcal{P} - \{P_1, P_2\}) \cup \{x_1 P_1 x''_1, x_2 P_2 x''_2\}$. Again, let P'_i be the component of $\mathcal{P}'$ containing x_i. The linkage $\mathcal{P}'$ satisfies the conclusions of the claim. We have proven that P'_1 and P'_4 have their termini in $S_1 \cap S_2$, and that no other path in $\mathcal{P}'$ has it's terminus in $S_1 \cap S_2$. Thus 2. and 3. follow. Finally, our original linkage $\mathcal{P}$ was such that x'_1 and x'_4 were not adjacent. When we consider the edge $e = uv \subseteq S_1 \cap S_2$, then at least one vertex of u and v must not be a terminus of a path in $\mathcal{P}'$, proving 4. Condition 5. holds by construction. □

We now want to show that if the two S_1 and S_2 intersect in four vertices, then the other S_i's can only intersect S_1 and S_2 in a very limited manner. Towards this, we prove the following claim.

Claim 6.6 *If $|S_1 \cap S_2| = 4$ and if S_3 satisfies $|S_3 \cap (S_1 \cup S_2)| \geq 3$, then $|S_3 \cap (S_1 \cup S_2)| = 3$ and $S_3 \cap (S_1 \cup S_2) \subseteq S_1 \cap S_2$.*

Proof Assume $|S_1 \cap S_2| = 4$ and $|S_3 \cap (S_1 \cup S_2)| \geq 3$. We know from Claim 6.5 that we have a linkage $\mathcal{P}$ with components $P_1, \ldots P_6$ from X to $S_1 \cup S_2$ with the path termini as described in the statement of Claim 6.5. Let x'_i be the terminus of P_i. Without loss of generality, assume x'_3 and x'_6 lie in $S_1 \cap S_2$, x'_1 and x'_2 lie in $S_1 - S_2$ and x'_4 and x'_5 lie in $S_2 - S_1$. Let the vertices w_1 and w_2 be the vertices of $S_1 \cap S_2$ that are not the termini of any path in $\mathcal{P}$. Notice that at least one of $w_1$and w_2 is an endpoint of the edge e, and so without loss of generality, we assume $w_1 \in S_3$. For notation, let $\mathcal{L}'$ be the linkage problem induced by $\mathcal{P}$ and $\mathcal{L}$ on $S_1 \cup S_2 - \{w_1, w_2\}$.

First assume at most one path P_i uses vertices of $B_3 - S_3$. Let y_1 and y_2 be the first and last vertices of P_i in S_3. Now there are two cases both of which are easily dealt with: either $S_3 \cap (S_1 \cup S_2) \subseteq S_1 \cap S_2$ or $S_3 \cap ((S_1 - S_2) \cup (S_2 - S_1)) \neq \varnothing$.

Case 1: $S_3 \cap (S_1 \cup S_2) \subseteq S_1 \cap S_2$.

If $|S_3 \cap S_1 \cap S_2| = 3$, then the claim is proven. Thus we may assume $|S_3 \cap S_1 \cap S_2| = 4$. Consequently, $x_3', x_6' \in S_3$. There exist paths Q_1, Q_2 in $B_3 - S_3$, where the ends of Q_1 are y_1 and y_2 and the ends of Q_2 are x_3' and x_6'.

Now consider the linkage $\mathcal{P}' = \mathcal{P} - \{P_i\} \cup \{x_i P_i y_1 Q_1 y_2 P_i x_i'\}$. This linkage is disjoint from the sets $B_1 - S_1$, $B_2 - S_2$, and $V(Q_2) - \{x_3', x_6'\}$. There exist disjoint paths with all internal vertices in $B_1 - S_1$ solving the linkage problem $\{\{x_1', w_1\}, \{x_2', w_2\}\}$. Similarly, there exist disjoint paths with all internal vertices in $B_2 - S_2$ solving the linkage problem $\{\{x_4', w_1\}, \{x_5', w_2\}\}$. Thus the linkage problem $\{\{x_1', x_4'\}, \{x_2', x_5'\}\}$ is feasible in $G[B_1 \cup B_2]$, and we contradict the fact that $\mathcal{L}$ is not feasible in G.

Case 2: $S_3 \cap ((S_1 - S_2) \cup (S_2 - S_1)) \neq \varnothing$.

Then some vertex of $\{x_1', x_2', x_4', x_5'\}$ lies in S_3. Without loss of generality, say $x_1' \in S_3$. Using the fact that $w_1 \in S_3$, we observe that there exist disjoint paths Q_1, Q_2 with all internal vertices in $B_3 - S_3$ where the ends of Q_1 are y_1 and y_2 and the ends of Q_2 are x_1' and w_1. As above, let $\mathcal{P}'$ be the linkage defined by $\mathcal{P} - \{P_i\} \cup \{x_i P_i y_1 Q_1 y_2 P_i x_i'\}$. There exist disjoint paths R_1, R_2 with all internal vertices in $B_1 - S_1$ where the ends of R_1 are x_3' and x_6' and the ends of R_2 are x_2' and w_2. There exist paths T_1, T_2 with all internal vertices in $B_2 - S_2$ where the ends of T_1 are x_4' and w_1 and the ends of T_2 are x_5' and w_2 respectively. We have the linkage:

$$x_1' Q_1 w_1 T_1 x_4', \qquad x_3' R_1 x_6', \qquad x_2' R_2 w_2 T_2 x_5'$$

solving the linkage problem $\mathcal{L}'$ avoiding any non-terminus vertex of $\mathcal{P}'$, a contradiction to the fact that $\mathcal{L}$ is not feasible in G.

The analysis of the cases above shows we may assume at least two paths $P_i, P_j \in \mathcal{P}$ use vertices of $B_3 - S_3$. Assume for the moment that P_i and P_j are the only paths using vertices of $B_3 - S_3$. We may assume that the two paths are not P_3 and P_6, otherwise we could simply link the first vertices of P_3 and P_6 in $B_3 - S_3$ and link the remaining pairs of terminals with paths in $B_2 - S_2$ and $B_1 - S_1$ meeting at the vertices w_1, w_2. Thus we may assume one of the paths P_1, P_2, P_4, P_5 intersects $B_3 - S_3$. Without loss of generality, say P_1. Let x_1'' be P_1's first vertex in S_3. Let P_i be the other path intersecting $B_3 - S_3$, and let y_1 and y_2 be the first and last vertices of P_i in S_3. There exist paths in Q_1, Q_2 in with all internal vertices in $B_3 - S_3$ where the ends of Q_1 are y_1 and y_2 and the ends of Q_2 are x_1'' and w_1. Let $\mathcal{P}'$ be the linkage defined by $P_i' = x_i P_i y_1 Q_1 y_2 P_i x_i'$ and $P_k' = P_k$ for $k \neq i$. There exist paths R_1 and R_2 with all internal vertices in $B_2 - S_2$ where the ends of R_1 are x_3' and x_6' and the ends of R_2 are x_2' and w_2. There exist paths T_1, T_2 with all internal vertices in $B_1 - S_1$ where the endpoints of T_1 are w_1 and x_4' and the endpoints of T_2 are x_5' and w_2. We get the following linkage:

$$x_1 P_1' x_1'' Q_2 w_1 T_1 x_4' P_4' x_4, \qquad x_2 P_2' x_2' R_2 w_2 T_2 x_5' P_5' x_5, \qquad x_3 P_3' x_3' R_1 x_6' P_6' x_6$$

that contradicts the fact that $\mathcal{L}'$ is not feasible.

Finally, three or more components of $\mathcal{P}$ cannot use vertices of $B_3 - S_3$, because each such path must use at least two vertices of S_3, and yet no $w_1 \in S_3 - V(\mathcal{P})$. This proves the claim. $\square$

We now will prove that if S_i and S_j intersect in four vertices, then S_i^* (and similarly in S_j^*), has three anti-edges not contained in any other S_k^*.

Claim 6.7 *For all distinct indices i, j, if S_i and S_j are such that $|S_i \cap S_j| = 4$, then each $G/e[S_i^*]$ and $G/e[S_j^*]$ have three anti-edges that they do not share with either other or any other $G/e[S_l^*]$.*

Proof For notation, assume that S_1 and S_2 are as in the statement of the claim and intersect in four vertices. Let $S_1 \cap S_2 = \{u, v, y_1, y_2\}$ where u and v are the endpoints of the edge e specified in the statement of the lemma. Let $\mathcal{P}$ be a linkage as in Claim 6.5 and let the components of $\mathcal{P}$ be labeled $P_1, \ldots, P_6$ such that P_i contains x_i. Let x_i' be the terminus of P_i in $S_1 \cup S_2$. Without loss of generality, assume $x_3', x_6' \in S_1 \cap S_2 = \{u, v, y_1, y_2\}$, and that x_1' and x_2' lie in $S_1 - S_2$. Let $\mathcal{L}'$ be the linkage problem on the appropriate subset of $S_1 \cup S_2$ induced by $\mathcal{L}$ and $\mathcal{P}$. Up to symmetry, there are two cases to consider: $\{x_3', x_6'\} = \{y_1, y_2\}$ or $\{y_1, v\}$.

Case 1: $\{x_3', x_6'\} = \{y_1, y_2\}$

By Claim 6.3, we know that each of S_1 and S_2 has some anti-edge not incident with e which is not contained in $S_1 \cap S_2$. Call them a_1 and a_2, respectively. Notice that by Claim 6.6, neither a_1 or a_2 can be contained in S_l^* for any $l \neq 1, 2$.

Consider what happens if u or v were adjacent to any vertex x_1', x_2', x_4', x_5'. Say v is adjacent to x_1'. Then there exist paths Q_1, Q_2 with all internal vertices $B_1 - S_1$ where the ends of Q_1 are x_2' and u and the ends of Q_2 are x_3' and x_6'. Also, there exist paths R_1, R_2 with all internal vertices in $B_2 - S_2$ such that the ends of R_1 are x_4' and v and the ends of R_2 are x_5' and u. We get the linkage

$$x_1' v R_1 x_4', \qquad x_2' Q_1 u R_2 x_5', \qquad x_3' Q_2 x_6'$$

proving that $\mathcal{L}'$ is solvable by paths not intersecting $V(\mathcal{P}) - \{x_1', \ldots, x_6'\}$, a contradiction.

Thus we may assume that no such edge exists and then u and v have no neighbor in $S_1 - S_2$ nor in $S_2 - S_1$. If we let v_e be the vertex in G/e coming from the edge e, then in which case, $a_1, (v_e, x_1'), (v_e, x_2')$ are three anti-edges contained in $G/e[S_1^*]$ that are not contained in any other S_l^*. If they were in some S_k^*, S_k would necessarily intersect $S_2 \cup S_1$ in at least three vertices and at least one vertex of $S_1 - S_2$, contrary to what we have seen above in Claim 6.6. Thus both $G/e[S_1^*]$ and $G/e[S_2^*]$ contain three anti-edges they do not share with each other or any other S_k^*.

Case 2: $\{x_3', x_6'\} = \{y_1, v\}$.

Again, as in the previous case, we may assume that neither y_2 nor u has any neighbor in $S_2 - S_1$ nor in $S_1 - S_2$. Thus if we consider $G[S_1]$, by Claim 6.3 applied to the vertices y_2 and u, there must exist at least two anti-edges not incident with y_2 or u. We conclude that there exists an anti-edge between either y_1 or v and one of either x'_1 and x'_2. Since $x'_1$and x'_2 are symmetric here, there are two distinct cases: x'_1 is not adjacent to v and x'_1 is not adjacent to y_1. If x'_1 is not adjacent to v, then the anti-edges $(v_e, x'_1), (x'_1, y_2), (x'_2, y_2)$ are contained in S^*_1. If x'_1 is not adjacent y_1, then S^*_1 contains the anti-edges $(x'_1, y_1), (x'_1, y_2)$, and (x'_2, y_2). In either case, $G[S^*_1]$ contains three anti-edges that cannot lie in any other S^*_l by Claim 6.6. This proves the claim. □

Our objective is to show that each S^*_i has at least three anti-edges not shared by any S^*_ℓ for $\ell \neq i$. We have just shown that if $|S_i \cap S_j| \geq 4$ for some $j \neq i$, then the three anti-edges may be chosen so that they belong to no other S^*_ℓ for $\ell \neq i$. To complete the proof let $i \in \{1, 2, \ldots, k\}$ be such that $|S_i \cap S_j| \leq 3$ for all $j \neq i$. If S_i has at least three anti-edges not incident with u or v, then those are clearly as required. Thus we may assume that S_i has at most two such anti-edges, and hence Claim 6.3 implies that it has exactly two and they share an end. Let those anti-edges be labeled $a^i_1 = (x^i, y^i)$ and $a^i_2 = (y^i, z^i)$, consistent with the notation in Claim 6.4. To complete the proof of Lemma 5.13 it suffices to show the following claim.

Claim 6.8 *If S_i and S_j are as above, then $z^i \neq z^j$.*

We may assume that $i = 1$ and $j = 2$. Suppose for a contradiction that $z^1 = z^2$. We show the linkage problem $\mathcal{L}$ is feasible, a contradiction. The intersection $S_1 \cap S_2 = \{u, v, z^1\}$, where u and v are the ends of e. We know that $a^1_1 \cap (S_1 \cap S_2) = \varnothing$.

Let $\mathcal{P}$ be the linkage in the statement of Claim 6.4. Let the components of $\mathcal{P}$ and the vertices of $S_1 \cup S_2$ be labeled such that the ends of $P_i \in \mathcal{P}$ are x_i and x'_i. Without loss of generality, assume that the termini of P_1 and P_4 form the anti-edge a^1_1. Let $\mathcal{L}'$ be the linkage problem induced by $\mathcal{L}$ and $\mathcal{P}$ on S_1. Three of the paths P_2, P_3, P_5, P_6 must have their ends in $S_1 \cap S_2$. Again without loss of generality, assume that $x'_3, x'_5, x'_6 \in S_1 \cap S_2$. We will separately consider the possible number of paths that utilize vertices of $B_2 - S_2$.

Case 1: no P_i contains vertices of $B_2 - S_2$.

Then there exists a path Q ends x'_3 and x'_6 and all internal vertices in $B_2 - S_2$. Then the linkage problem $\{\{x'_1, x'_4\}, \{x'_2, x'_5\}\}$ is feasible in $G[B_1]$, implying that a solution to the linkage problem $\mathcal{L}'$ exists with no internal vertex intersecting $\mathcal{P}$, a contradiction.

Case 2: exactly one path $P_i \in \mathcal{P}$ contains vertices of $B_2 - S_2$.

Let w_1 be the vertex of P_i in S_2 closest to X on P_i, and w_2 be the vertex of P_i in S_2 closest to S_1 on P_i. There exist paths Q_1 and Q_2 with all internal vertices in $B_2 - S_2$ such that the ends of Q_1 are w_1 and w_2 and the ends of Q_2 are x'_3 and x'_6. Then the linkage $\mathcal{P}' = \mathcal{P} - \{P_i\} \cup \{x_i P_i w_1 Q_1 w_2 P_i x'_i\}$ has the same endpoints as $\mathcal{P}$. We can link x'_3 and x'_6 avoiding all other vertices of $\mathcal{P}'$. As in the previous case, the fact that the linkage problem $\{\{x'_1, x'_4\}, \{x'_2, x'_5\}\}$ is feasible in $G[B_1]$ implies that $\mathcal{L}$ is feasible, a contradiction.

Case 3: exactly two paths $P_i, P_j \in \mathcal{P}$ contain vertices of $B_2 - S_2$.

First, assume $i = j + 3$ or $j = i + 3$. Then we can link x_i to x_j with a path in B_2 avoiding the other paths of $\mathcal{P}$. The other two pairs of vertices in $\mathcal{L}$ can be linked in $G[B_1]$, implying that $\mathcal{L}$ is feasible.

Thus we conclude $i \neq j + 3$ and $j \neq i + 3$. Now assume $i = 3$. Let x_3'' be the vertex of S_2 closest to x_3 on P_i. Let w_1 and w_2 be the vertices of S_2 on P_j closest to x_j and x_j' on P_j, respectively. Then there exist paths Q_1 and Q_2 with all internal vertices in $B_2 - S_2$ such that the ends of Q_1 are x_3'' and x_6' and the ends of Q_2 are w_1 and w_2. Then let $P' = x_j P_j w_1 Q_2 w_2 P_j x_j'$ and $P_k' = P_k$ for $k \neq j$.. The ends of P_k' are equal to the ends of P_k for all indices k. There exist paths R_1 and R_2 with all internal vertices in $B_1 - S_1$ where the ends of R_1 are x_1' and x_4' and the ends of R_2 are x_2' and x_5'. Then we have the linkage

$$x_1 P_1' x_1' R_1 x_4' P_4' x_4, \qquad x_2 P_2' x_2' R_2 x_5' P_5' x_5, \qquad x_3 P_3' x_3'' Q_1 x_6' P_6' x_6$$

solving the linkage problem $\mathcal{L}$, a contradiction.

We conclude that $i \neq 3$, and symmetrically, $i, j \neq 6$. Then at least one of i or j is equal to one or four. Without loss of generality, assume $i = 4$. Then P_i must use two vertices of $S_2 - S_1$. It follows that $j = 5$ since $x_5' \in S_1 \cap S_2$. Let x_5'' be the unique vertex of P_5 in $S_2 - S_1$ and w_1 the vertex of P_4 in $S_2 - S_1$ closest to x_4 on P_4 and w_2 the other vertex of P_4 in $S_2 - S_1$. There exist disjoint paths Q_1 and Q_2 with all internal vertices in $B_2 - S_2$ such that the ends of Q_1 are x_5'' and w_2 and the ends of Q_2 are w_1 and x_5'. There exist disjoint paths R_1 and R_2 with all internal vertices in $B_1 - S_1$ such that the ends of R_1 are x_3' and x_6' and the ends of R_2 are x_5' and x_1'. Notice by the fact that a_1^1 is the anti-edge (x_1', x_4') and the second anti-edge in $G[S_1]$ not incident to e must have z^1 as an endpoint, we conclude that x_2' is adjacent to x_1' and x_4'. The linkage

$$x_1 P_1 x_1' R_2 x_5' Q_2 w_1 P_4 x_4, \quad x_2 P_2 x_2' x_4' P_4 w_2 Q_1 x_5'' P_5 x_5, \quad x_3 P_3 x_3' R_1 x_6' P_6 x_6$$

contradicts the fact that $\mathcal{L}$ is not feasible.

Case 4: exactly three paths in $\mathcal{P}$ contain vertices of $B_2 - S_2$

Each of these paths must use at least two vertices in S_2. Since P_3, P_5, P_6 each must use one vertex of S_2, it follows that each of P_3, P_5, P_6 uses vertices of $B_2 - S_2$, and each one uses exactly one vertex of $S_2 - S_1$. Let x_3'', x_5'', x_6'' be the vertices of P_3, P_5, P_6 respectively in $S_2 - S_1$. Then there exists paths Q_1, Q_2 with all interior vertices in $B_2 - S_2$ where the ends of Q_1 are x_3'' and x_6'' and the ends of Q_2 are x_5'' and x_5'. There exist paths R_1 and R_2 with all internal vertices in $B_1 - S_1$ where the ends of R_1 are x_1' and x_4' and the ends of R_2 are x_2' and x_5'. The linkage

$$x_1 P_1 x_1' R_1 x_4' P_4 x_4, \qquad x_2 P_2 x_2' R_2 x_5' Q_2 x_5'' P_5 x_5, \qquad x_3 P_3 x_3'' Q_1 x_6'' P_6 x_6$$

contradicts the fact that $\mathcal{L}$ is not feasible.

This completes the proof of the claim. $\square$

Now we have completed the proof of Lemma 5.13. We have shown that for each S_i, upon contracting the edge e, $G/e[S_i^*]$ contains at least three anti-edges not contained in any other S_j^* implying that if we sum over every such S_i, there is a total of at least $3k$ anti-edges contained in $\bigcup_i G/e[S_i^*]$, as desired. $\square$

7 Lower Bounds

In [13], we conjecture that if a graph is $2k$ connected, has n vertices, and $(2k-1)n - \frac{k}{2}(3k+1)+1$ edges, then the graph is k-linked. There is an infinite family of graphs showing this would be the optimal edge bound for every $k \geq 2$. In the interest of completeness, we present the example appearing in [13] showing that the bound in Theorem 1.1 is optimal. Let P_1, P_2, P_3, P_4 be four paths on k vertices with the vertices of P_i labeled $x_1^i, x_2^i, \ldots, x_k^i$. Let u_1, u_2, y_5, y_6 be four additional vertices. Then let the edges of G be defined thus, where the superscript addition is taken mod four

$$\begin{aligned} E(G) =& \{x_j^i x_j^{i+1} : j = 1, \ldots, k, i = 1, \ldots, 4\} \cup \\ & \cup \{x_j^i x_{j+1}^{i+1} : j = 1, \ldots k-1, i = 1, \ldots, 4\} \cup \\ & \cup \{u_i x_k^j : i = 1, 2, j = 1, \ldots, 4\} \cup \\ & \cup \{y_i x_j^l : i = 5, 6, j = 1, \ldots, k, i = 1, \ldots 4\} \cup \\ & \cup \{u_1 u_2, y_1 u_1, y_1 u_2, y_2, u_1, y_2 u_2\} \end{aligned}$$

Then G is 6-connected and $|E(G)| = 5|V(G)| - 15$, but the linkage problem $\{\{x_1^1, x_1^3\}, \{x_1^2, x_1^4\}, \{y_1, y_2\}\}$ is not feasible, implying that G is not 3-linked.

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