Packing non-zero A-paths in an undirected model of group labeled graphs

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Abstract

Let Γ be an abelian group, and let $\gamma : E(G) \to \Gamma$ be be a function assigning values in Γ to every edge of a graph G. For a subgraph H of G, let $\gamma(H) = \sum_{e \in E(H)} \gamma(e)$. For a set A of vertices of G, an A-path is a path with both endpoints in A and otherwise disjoint from A. In this article, we show that either there exist k vertex disjoint A-paths $P_1, P_2, \dots, P_k$ such that $\gamma(P_i) \neq 0$ for all $1 \leq i \leq k$, or there exists a set X of vertices such that $G - X$ does not contain a non-zero A-path with $|X| \leq 50k^4$.

Key Words : Group labeled graphs, Disjoint paths, A-paths

1 Introduction

Given a graph G and a set $A \subseteq V(G)$, an *A-path* is a path with both endpoints laying in A and no internal vertex in A. Much work has gone into determining when a given graph contains many disjoint A-paths satisfying some specified property. Gallai proved in [2] that a given graph G with a specified set A of vertices either has k disjoint A-paths or there exists a set of at most $2k-2$ vertices hitting every A-path. Mader [4] generalized this result as follows. Let G be a graph with a specified set A of vertices and let $\mathcal{S}$ be a partition of the set A. Mader showed that either there exist k disjoint A-paths $P_1, \dots, P_k$ such that P_i has endpoints in distinct sets of the partition $\mathcal{S}$, or there exists a set of $2k-2$ vertices hitting all such A-paths. See [5] for a short proof of this result. In each case, the bound on the hitting set is the best possible and actually comes from an exact min-max theorem for the number of such paths. Kriesell [3] proved that a similar min-max result holds for directed A-paths in digraphs.

In this article, we will utilize two distinct models of group labeled graphs. The first introduced here will be our primary focus.

Definition Let Γ be an abelian group and G a graph. An *undirected Γ-labeling of G* is a function $\gamma : E(G) \to \Gamma$. For a subgraph H of G, let $\gamma(H) = \sum_{e \in E(H)} \gamma(e)$ be the *weight* of the subgraph H.

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We will see that either a given undirected Γ- labeled graph contains many disjoint non-zero A-paths or there exists a set of vertices of bounded size hitting all such A-paths. The following theorem is the main result of this article.

Theorem 1.1 *Let Γ be an abelian group and let γ be an undirected Γ-labeling of a graph G. Let $A \subseteq V(G)$ be a set of vertices of G. Then for all integers $k \geq 1$, either G contains k pair-wise vertex disjoint A-paths $P_1, \ldots, P_k$ such that $\gamma(P_i) \neq 0$ for all $i = 1, \ldots, k$, or there exists a set $X \subseteq V(G)$ with $|X| \leq 50k^4$ such that every A-path P in $G - X$ has $\gamma(P) = 0$.*

An immediate corollary of Theorem 1.1 is the following.

Corollary 1.2 *Let G be a graph and let m be a positive integer. Let A be a fixed set of vertices. Then for all integers $k \geq 1$, either there exists k disjoint A-paths $P_1, \ldots, P_k$ such that the length of P_i is not congruent to 0 mod m, or there exists a set X of vertices with $|X| \leq 50k^4$ such that in $G - X$ every A-path has length congruent to zero mod m.*

In recent work, Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour [1] generalize the results of Gallai and Mader by considering a different definition of group labeled graphs. In this model of group labeled graphs, the edges are assigned an orientation as well as a group value. When calculating the weight of a path, the weight will be added if the edge is traversed in the same direction as the orientation, and subtracted if it is traversed contrary to the orientation. Explicitly, we give the following definition. In this group labeling of the graph, the group need not be abelian and we use multiplicative notation.

Definition Let Γ be an arbitrary group and G a graph. A *oriented Γ-labeling of G* is a pair of functions (γ, dir) satisfying the following. The function $\gamma : E(G) \rightarrow \Gamma$ maps $E(G)$ to Γ. The function dir is an orientation of the edges defined $dir : \{(x, y) \in V(G) \times V(G) : xy \in E(G)\} \rightarrow \{1, -1\}$ such that $dir(u, v) = -dir(v, u)$ for all edges uv in $E(G)$. Let P be a path in G and let the vertices of the path be $v_1, v_2, \ldots, v_k$ with v_i adjacent v_{i+1} for $1 \leq i \leq k - 1$. Then P is a *non-zero path* if $\prod_{i=1}^{k-1} \gamma(v_i v_{i+1})^{dir(v_i, v_{i+1})}$ is not equal to the identity in Γ.

Observe when calculating $\prod_{i=1}^{k-1} \gamma(v_i v_{i+1})^{dir(v_i, v_{i+1})}$ for a a given path P in an oriented group labeled graph, the exact value will typically depend on which end of the path is labeled to be v_1. However, whether or not $\gamma(P)$ is equal to the identity is independent of the direction in which the vertices of the path are traversed, and so non-zero paths are in fact well defined.

We recall that a non-identity element α of a group Γ is of *order two* if $\alpha = -\alpha$. If Γ is an abelian group such that every element of Γ is of order two, then the two different models of group labeled graphs coincide since whether an edge is traversed according to the orientation or contrary to it, the same value α will be added. We will make use of the following observation formalizing this idea.

Observation 1 *Let G be a graph and Γ an abelian group. Let $\gamma : E(G) \rightarrow \Gamma$ be any function. Let $A \subseteq V(G)$ and P be an A-path in G. If for every edge e of P, $\gamma(e)$ is either equal to zero or an element of Γ of order two, then for all orientations dir of the edges of G, the weight of P in in the oriented Γ-labeling (γ, dir) is equal to the weight in the undirected labeling γ.*

Chudnovsky et al. prove the following theorem.

Theorem 1.3 ([1]) *Let Γ be a group, let G be a graph, and let γ and dir be two functions such that (γ, dir) is an oriented Γ-labeling of G. Let A be a specified set of vertices in G. Then either*

1. *there exist k vertex disjoint non-zero A-paths, or*
2. *there exists a set X of at most $2k-2$ vertices such that $G-X$ contains no non-zero A-path.*

In fact, the authors demonstrate an exact min-max result for the number of such non-zero paths which immediately implies Theorem 1.3. By choosing an appropriate group labeling, Theorem 1.3 implies the min-max results of both Mader and Gallai mentioned above.

We first establish definitions to discuss paths and collections of paths contained in a larger graph. Let P be a path with endpoints x and y and let z and z' be two vertices of P. Then by zPz', we refer to the subpath of P containing z and z'.

Definition A *linkage* is a graph $\mathcal{P}$ where every connected component is a path.

A connected component of a linkage $\mathcal{P}$ is a *composite path* of the linkage. In a slight abuse of notation, we will sometimes refer to the path $P \in \mathcal{P}$ for a linkage $\mathcal{P}$ to mean that P is a composite path of $\mathcal{P}$. Given a linkage $\mathcal{P}$ contained as a subgraph in a larger graph G, a *$\mathcal{P}$-bridge* is either an edge of $G - E(\mathcal{P})$ with both ends contained in $V(\mathcal{P})$ or a component C of $G - V(\mathcal{P})$ along with any edges with one endpoint in $V(C)$ and one endpoint in $V(\mathcal{P})$. A $\mathcal{P}$-bridge is *trivial* if it consists of a single edge. Given a $\mathcal{P}$-bridge B, the vertices of $V(B) \cap V(\mathcal{P})$ are the *attachments* of B. A bridge B is *stable* if it has attachments on two distinct components of $\mathcal{P}$.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 proceeds in two steps. We define an A-star as follows.

Definition Let G be a graph and $A \subseteq V(G)$. Let $P_1, \dots, P_l$ be A-paths. If there exists a vertex $v \in V(G) - A$ such that $V(P_i) \cap V(P_j) = v$ for every $1 \le i < j \le l$, then the subgraph consisting of $P_1 \cup P_2 \cup \cdots \cup P_l$ is an *A-star*. The paths $P_1, \dots, P_l$ are the *composite paths* of the A-star. The vertex v shared by every path P_i is the *nexus* of the A-star. For each composite path P_i, if x is an end of P_i in A, the subpath xP_iv is a *ray* of the A-star.

Let γ be a Γ-labeling of a graph G and let A be a fixed set of vertices. An A-star $\mathcal{S}$ with composite paths $P_1, \dots, P_n$ is a *non-zero A-star* if $\gamma(P_i) \neq 0$ for all $i = 1, \dots, n$. Notice that the choice of composite paths for a given non-zero A-star is not necessarily unique. We will show that in any group labeled graph for any set A of vertices, either there exists a non-zero A-star with l composite paths, or there exists k disjoint non-zero A-paths, or there exists a set X of bounded size, hitting every non-zero A-path. Rigorously, we state this as the following lemma.

Lemma 2.1 *Let Γ be an abelian group G a graph. Let γ be a Γ-labeling of G, and let $A \subseteq V(G)$. Let the function f_1 be defined as follows:*

$$\begin{aligned} f_1(k,l) &:= 2(2kl + 3k) - 2 + (2k-2) \\ &= 4k(l+2) - 4. \end{aligned}$$

Then for all positive integers k and l, either there exists a set X of vertices with $|X| \le f_1(k,l)$ such that every A-path in $G - X$ has zero weight, or there exists a non-zero A-star with l composite paths, or there exists k vertex disjoint A-paths $P_1, \dots, P_k$ such that $\gamma(P_i) \neq 0$ for all $i = 1, \dots, k$.

Notice that Lemma 2.1 implies Theorem 1.1 when the graph is assumed to have bounded degree.

Corollary 2.2 *Let Γ be an abelian group and (G, γ) a Γ-labeled graph with $\Delta(G) = d$. Let A be a subset of vertices of G. Then either there exist k vertex disjoint A-paths each with non-zero weight, or there exists a set X of at most $f_1(k, \lfloor d/2 \rfloor + 1)$ vertices such that every A-path in $G - X$ has zero weight.*

In the proof of Theorem 1.1, we will in fact find k distinct A-stars where each composite path has non-zero weight, such that, while not disjoint, at least each A-star has a unique nexus vertex. Moreover, we will see that we can choose these A-stars to have as many composite paths as we will need. The second step in the proof of Theorem 1.1 is to show that it is possible to "uncross" these non-zero A-stars to find k vertex disjoint non-zero A-stars, at the expense of sacrificing some of the composite paths of the original A-stars.

Lemma 2.3 *Let Γ be an abelian group and G a graph. Let γ be a Γ-labeling of G. Let A be a set of vertices of G, and let $k, t,$ and l be positive integers. Let*

$$n = t[f_1(k, t + 1)] + 8tl + (t + l)$$

Let $\mathcal{S}_1, \ldots, \mathcal{S}_t$ be a collection of non-zero A-stars each with n composite paths. For all $i = 1, \ldots, t$, let $P^i_1, \ldots, P^i_n$ be non-zero composite paths of $\mathcal{S}_i$ and let v_i be the nexus vertex of $\mathcal{S}_i$. Furthermore, assume that $v_i \neq v_j$ for all $i \neq j$. Then either G contains k pair-wise vertex disjoint A-paths $Q_1, \ldots, Q_k$ such that $\gamma(Q_i) \neq 0$ for all $i = 1, \ldots, k$, or there exists a collection of A-stars $\overline{\mathcal{S}_1}, \ldots, \overline{\mathcal{S}_t}$ such that the following hold:

1. *$\bigcup_{i=1}^t V(\overline{\mathcal{S}_i}) \subseteq \bigcup_{i=1}^t V(\mathcal{S}_i)$,*
2. *For all indices $i = 1, \ldots, t$, $\overline{\mathcal{S}_i}$ has l composite paths $\overline{P}^i_1, \ldots, \overline{P}^i_l$, and furthermore, $\gamma(\overline{P}^i_j) \neq 0$ for all $j = 1, \ldots, l$.*
3. *For every pair of distinct indices $1 \leq i, j \leq t$, $V(\overline{\mathcal{S}_i}) \cap V(\overline{\mathcal{S}_j}) = \emptyset$.*

We now see that Theorem 1.1 follows easily assuming the lemmas.

Proof. *(Theorem 1.1, assuming Lemmas 2.1 and 2.3)*

Let G be a graph and Γ an abelian group. Let γ be an undirected Γ-labeling of G. Let A be a fixed subset of the vertices of G. The theorem is trivially true when $k = 1$, thus we let $k \geq 2$ be a positive integer. Set

$$\begin{aligned} m :&= k\,[f_1(k, k + 1)] + 9k + 1 \\ &= k(4k(k + 3) - 4) + 9k + 1 \\ &\leq 12k^3. \end{aligned}$$

We let

$$\begin{aligned} n :&= k + f_1(k, m) \\ &\leq k + 4k(12k^3 + 2) - 4 \\ &\leq 50k^4. \end{aligned}$$

We may assume that G does not contain k disjoint non-zero A-paths. By our choice of n, we claim that either there exists a set X of at most n vertices hitting every non-zero A-path, or there exist non-zero A-stars $S_1, S_2, \ldots, S_k$ each with m composite paths. Moreover, if v_i is the nexus vertex of S_i, then $v_i \neq v_j$ for $i \neq j$. The existence of one such A-star follows immediately from Lemma 2.1. Given i such A-stars $S_1, \ldots, S_i$ for $i < k$, consider the graph $G - \{v_1, v_2, \ldots, v_i\}$. Again, by applying Lemma 2.1, there either exists a non-zero A-star with m composite paths, or there exists a set X of size at most $f_1(k, m)$ hitting all non-zero A-paths in $G - \{v_1, \ldots, v_i\}$. In the first case, we find the A-star S_{i+1} with nexus vertex disjoint from $\{v_1, \ldots, v_i\}$, and in the second case, the set $X \cup \{v_1, \ldots, v_i\}$ intersects every non-zero A-path in G and has size at most $f_1(k, m) + i \leq n$.

Given such non-zero A-stars $S_1, \ldots, S_k$, by our choice of m and Lemma 2.3 there exist k vertex disjoint non-zero A-stars each containing one composite path. In other words, we find k disjoint non-zero A-paths and the theorem is proven. ■

3 Bridges in group labeled graphs

A classic theorem of Tutte [7] states that given a linkage $\mathcal{P} = P_1 \cup P_2 \cup \cdots \cup P_t$ contained in a 3-connected graph G, there exists a linkage $\mathcal{P}' = P'_1 \cup P'_2 \cup \cdots \cup P'_t$ where P_i and P'_i have the same endpoints and furthermore every $\mathcal{P}'$-bridge is stable. We will need a similar result for group labeled graphs. However, difficulties arise since rerouting a given linkage to ensure every bridge is stable may destroy valuable properties concerning the weights of the paths in the linkage.

Let $\mathcal{P}$ be a linkage in a Γ-labeled graph and let P be a connected component of $\mathcal{P}$ with endpoints u and v. Let γ be the corresponding weight function. If $\gamma(P) = 0$, a vertex $x \in V(P)$ is a *breaking vertex* if $\gamma(vPx) = \alpha \neq 0$ and furthermore, α is not of order two in the group Γ.

We now see sufficient conditions to ensure that we can find a linkage with every non-trivial component of weight zero has a stable bridge attaching to a breaking vertex. We recall that a *separation* in a graph G is a pair (X, Y) with $X \subsetneq V(G)$, $Y \subsetneq V(G)$ such that every edge xy of G either satisfies $x, y \in X$ or $x, y \in Y$. In other words, no edge xy of G has $x \in X - Y$ and $y \in Y - X$. The *order* of a separation (X, Y) is $|X \cap Y|$.

Lemma 3.1 *Let G be a Γ- labeled graph with weight function γ. Let $\mathcal{P}$ be a linkage with composite paths $P_1, \ldots, P_k$ with the ends of P_i labeled x_i and y_i. We allow $\mathcal{P}$ to contain trivial paths P_i in which case $x_i = y_i$. Let $X = \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. Assume that for every separation (A, B) with $X \subseteq A$ the order of (A, B) is at least two, and if the order of (A, B) equals two, then there exist paths R_1 and R_2 in $G[B]$ such that the endpoints of R_i lie in $A \cap B$ and furthermore $\gamma(R_1) \neq \gamma(R_2)$.*

If every non-trivial composite path P of $\mathcal{P}$ either has a breaking vertex or satisfies $\gamma(P) \neq 0$, then there exists a linkage $\mathcal{P}'$ with composite paths $P'_1, \ldots, P'_k$ such that the following hold:

1. *the endpoints of P'_i are x_i and y_i,*
2. *if i is such that P_i is a non-trivial path and the weight $\gamma(P_i) \neq 0$, then $\gamma(P'_i) \neq 0$, and*
3. *if i is such that P_i is non-trivial and $\gamma(P'_i) = 0$, then there exists a stable $\mathcal{P}'$-bridge B and a breaking vertex x of P'_i such that B has x as an attachment.*

Proof. The proof will proceed by carefully selecting a potential counter-example and deriving a contradiction. We begin with three desirable properties that we will require when we choose a potential counter-example to the lemma. However, before we fix a counter-example and proceed with the proof, we first present several implications for a linkage $\mathcal{P}'$ satisfying properties A, B, and C.

A. The linkage $\mathcal{P}'$ satisfies 1. and 2. and for every non-trivial composite path P_i' with $\gamma(P_i') = 0$, the path P_i' contains a breaking vertex.

B. Subject to A, the number of non-trivial composite paths violating 3. is minimized.

C. Subject to A, and B, the number of vertices contained in stable bridges is maximized.

Notice $\mathcal{P}$ is a linkage satisfying A, implying that such a choice of $\mathcal{P}'$ exists.

We begin with several preliminary observations.

Claim 3.2 *Let $\mathcal{P}'$ be a linkage satisfying A, B, and C. with composite paths $P_1', \dots, P_k'$. For any non-trivial composite path P_i' of $\mathcal{P}'$ that violates condition 3., there do not exist a separation (W_1, W_2) of G with $X \subseteq W_1$ and $W_1 \cap W_2 \subseteq P_i'$.*

Proof. Assume otherwise, and let (W_1, W_2) be such a separation. Let u and v be the vertices of $W_1 \cap W_2$, and assume u is the closer to x_i in P_i'. By our assumptions on G, in the subgraph $G[W_2]$, there exist two paths R_1 and R_2 linking u and v with $\gamma(R_1) \neq \gamma(R_2)$. For either $j = 1$ or 2, the path $x_i P_i' u R_j v P_i' y_i$ must have non-zero weight. It follows that the linkage $\mathcal{P}' - P_i' \cup x_i P_i' u R_j v P_i' y_i$ violates our choice of $\mathcal{P}'$ to minimize the number of non-trivial paths violating condition 3. for some value of $j = 1, 2$. ■

Claim 3.3 *Let $\mathcal{P}'$ be a linkage satisfying A., B. and C. with composite paths $P_1', \dots, P_k'$. Let z_i be a breaking vertex of a path P_i' violating condition 3. Let B_1 be a stable bridge and B_2 a non-stable bridge attaching to P_i'. Then there do not exist distinct vertices u, s_1 and s_2 where u is an attachment of B_1 and s_1, s_2 attachments of B_2 such that the vertices s_1, u, s_2, z_i occur on P_i' in that order (with the vertex s_2 possibly equal to the vertex z_i).*

Proof. Assume the claim is false and let $\mathcal{P}'$, P_i', B_1, B_2, s_1, s_2, u, and z_i be as in the statement. There exists a path R in B_2 with endpoints s_1 and s_2 and otherwise disjoint from P_i'. Let the endpoints of P_i' be x_i and y_i and assume the vertices s_1 is the closer of s_1 and s_2 to the vertex x_i on P_i'. The linkage $(\mathcal{P}' - P_i') \cup x_i P_i' s_1 R s_2 P_i' y_i$ contradicts our choice of $\mathcal{P}'$. To see this, observe first that $\gamma(x_i P_i' s_1 R s_2 P_i' y_i) = \gamma(P_i') = 0$ by the fact that $\mathcal{P}'$ satisfies B. It follows that z_i is a breaking vertex of $x_i P_i' s_2 R s_1 P_i' y_i$. Yet the vertex u is now an internal vertex of a stable bridge, contradicting our choice of $\mathcal{P}'$ to satisfy C. This completes the proof of the claim. ■

We will make several further refinements before picking a potential counter-example to Lemma 3.1. Towards that end, we define the following special vertices. Let $\mathcal{P}'$ be any linkage satisfying A, B, and C. with components $P_1', \dots, P_k'$. Let i be an index such that P_i' is nontrivial and $\gamma(P_i') = 0$, but P_i' violates condition 3. Let z_i be a breaking vertex in P_i'. Let $u(z_i)$ be an attachment of a stable bridge on the subpath $x_i P_i' z_i$ chosen as close to z_i as possible on the subpath $x_i P_i' z_i$. A bridge B *straddles* a vertex v in a path P if B has attachments in both components of $P - v$. Let $v(z_i)$ be the attachment in $z_i P_i' y_i$ of either

i. a stable bridge, or

ii. a bridge straddling both the vertices $u(z_i)$ and z_i,

with $v(z_i)$ chosen to be as close as possible to z_i as possible on the subpath $z_i P'_i y_i$.

Claim 3.4 *There exists a linkage $\mathcal{P}'$ satisfying A, B, and C. with components $P'_1, \ldots, P'_k$ such that for every index i such that P'_i violates condition 3, there exists a breaking vertex z_i such that both $u(z_i)$ and $v(z_i)$ are defined.*

Proof. Let P' be a linkage satisfying A, B, and C. with components $P'_1, \ldots, P'_k$. Let i be an index such that P'_i fails to satisfy condition 3. By Claim 3.2, the endpoints x_i and y_i of P'_i do not separate the vertices of P'_i from the remaining paths of $\mathcal{P}'$. Consequently, some stable bridge attaches to an internal vertex of P'_i. Moreover, no stable bridge attaches to a breaking vertex of P'_i. By possibly re-labeling the endpoints of P'_i, we may assume that there exists a breaking vertex z_i on P'_i such that $u(z_i)$ is defined.

Let $\mathcal{P}'$ satisfy A, B, and C. and let z_i be a breaking vertex on P'_i for every component P'_i of $\mathcal{P}'$ that fails to satisfy condition 3. such that $u(z_i)$ is defined. The linkage $\mathcal{P}'$ is the linkage desired by the claim. Let i be an index such that P'_i fails to satisfy condition 3. Again, by Claim 3.2, the vertices $u(z_i)$ and y_i do not form a 2-cut separating $u(z_i)P_i y'_i$ from $\mathcal{P}' - u(z_i)P'_i y_i$. It follows that there exists a bridge B attaching to an internal vertex of $u(z_i)P'_i y_i$ and with a second attachment in $(\mathcal{P}' - P'_i) \cup (z_i P'_i u(z_i) - \{u(z_i)\})$. If B is a stable bridge, then B cannot have an attachment in the subpath $u(z_i)P'_i z_i$ by our choice of $u(z_i)$, and if B has an attachment on an internal vertex of $z_i P'_i y_i$, the vertex $v(z_i)$ is defined and the claim is proven. It follows that we may assume that B is not a stable bridge. If B has as an attachment an internal vertex of $z_i P'_i y_i$, then B straddles both z_i and $u(z_i)$ and again the vertex $v(z_i)$ is defined. Thus we may assume that B has an attachment an internal vertex s_1 of the subpath $u(z_i)P'_i z_i$ and a vertex s_2 in the subpath $x_i P'_i u(z_i) - \{u(z_i)\}$. This contradicts Claim 3.3, completing the proof the claim. ■

We are now ready to pick a counter-example to Lemma 3.1. Let $\mathcal{P}'$ be a linkage satisfying A, B, and C. For every index i such that P'_i violates condition 3. we fix a breaking vertex z_i. Furthermore, we assume

D. For every index i such that P'_i violates condition 3, the vertices $u(z_i)$ and $v(z_i)$ are defined.

E. Subject to A, B, C, and D. the $\sum_{\{i:P'_i \text{ violates } 3.\}} |V(u(z_i)P'_i v(z_i))|$ is minimized.

Fix an index i such that P'_i violates condition 3. To simplify the notation, for the remainder of the proof we set $u(z_i) = u_i$ and $v(z_i) = v_i$.

As a case, assume there exists a stable bridge attaching to v_i. The vertices u_i and v_i do not form a 2-cut in G, and so we see that there must exist a bridge B attaching to both $u_i P'_i v_i - \{u_i, v_i\}$ and $\mathcal{P}' - u_i P'_i v_i$. The bridge B cannot be a stable bridge by our choice of u_i and v_i to be as close as possible to the vertex z_i on P'_i. There are two symmetric cases when the bridge B has an attachment in $u_i P'_i z_i - \{u_i\}$ or alternatively an attachment in $z_i P'_i v_i - \{v_i\}$. Assume the former. If B straddles the vertex u_i, then there exist attachments s_1 and s_2 of B such that the vertices s_1, u_i, s_2, z_i occur on the path P'_i in that order, contradicting Claim 3.3. Alternatively, the bridge B must straddle both the vertices z_i and v_i. By flipping the labels x_i and y_i, we violate our choice of u_i and v_i to satisfy E.

We conclude that there exists a non-stable bridge B' attaching at the vertex v_i straddling both z_i and u_i. Note that by our choice of v_i, the bridge B' has no attachments to an internal vertex of the subpath $z_iP_i'v_i$ and by Claim 3.3, the bridge B' has no attachments in $u_iP_i'z_i - \{u_i\}$. Let s_1 be an attachment of B' in the subpath $x_iP_i'u_i - \{u_i\}$ and let R_1 be a path linking s_1 to v_i in B'. The vertices u_i and v_i do not form a 2-cut in G, so there must exist a third bridge B'' attaching to an internal vertex of $u_iP_i'v_i$ and attaching to a vertex of $\mathcal{P}' - u_iP_i'v_i$. The bridge B'' cannot be a stable bridge by our choice of u_i and v_i to be as close as possible to the vertex z_i. There are now essentially four cases to consider: the bridge B'' may have one attachment in either $u_iP_i'z_i$ or $z_iP_i'v_i$ and a second attachment in either $x_iP_i'u_i - \{u_i\}$ or $v_iP_i'y_i - \{v_i\}$.

We first consider what happens when the bridge B'' attaches to a vertex s_2 of $z_iP_i'v_i - \{z_i, v_i\}$. The bridge B'' cannot attach to the subpath $x_iP_i'u_i$ by our choice of v_i to be as close to z_i as possible. Thus B'' has an attachment s_3 in the subpath $v_iP_i'y_i - \{v_i\}$. Let R_2 be a path in B'' linking s_2 and s_3. Consider the linkage $(\mathcal{P}' - P_i) \cup x_iP_i's_2R_2s_3P_i'y_i$. By B., the path $x_iP_i's_2R_2s_3P_i'y_i$ must have weight zero. It follows that z_i is a breaking vertex of $x_iP_i's_2R_2s_3P_i'y_i$. Moreover, the bridge containing the path R_1 attaches at the vertex s_2 of the path $x_iP_i's_2R_2s_3P_i'y_i$, contradicting our choice of $\mathcal{P}'$ to satisfy E.

We now consider the case when B'' has an attachment s_2 in $u_iP_i'z_i - \{u_i\}$. If B'' attaches to a vertex s_3 of $x_iP_i'u_i - \{u_i\}$, we contradict Claim 3.3. Thus we may assume B'' has an attachment s_2 in $u_iP_i'z_i - \{u_i\}$ and an attachment s_3 in $v_iP_i'y_i - \{v_i\}$. Let R_2 be a path linking s_2 and s_3 in B''. See Figure 1. Let Γ' be the subgraph of Γ consisting of 0 and all elements of Γ of order two.

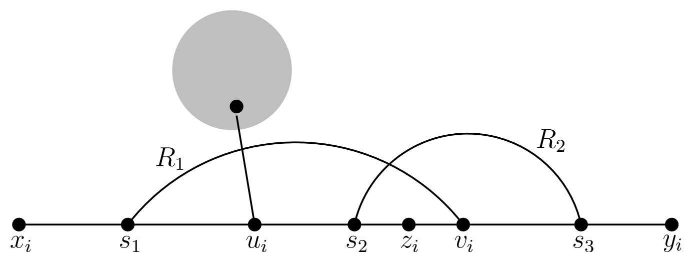

Figure 1: Rerouting the path P_i' using the paths R_1 and R_2 in the case when s_2 lies in $u_iP_i'z_i - \{u_i\}$ and s_3 lies in $v_iP_i'y_i - \{v_i\}$.

We first observe that $\gamma(x_iP_i's_1)$ and $\gamma(v_iP_i'y_i)$ are both contained in Γ' since both s_1 and v_i are an attachment of a stable bridge of $(\mathcal{P}' - P_i') \cup x_iP_i's_1R_1v_iP_i'y_i$ and we chose $\mathcal{P}'$ to satisfy B. Also by our choice of $\mathcal{P}'$ to satisfy B, we see that $\gamma(R_1) \in \Gamma'$ since $\gamma(x_iP_i's_1R_1v_iP_i'y_i) = 0$. However, by the fact that z_i is a breaking vertex, $\gamma(z_iP_i'y_i) \notin \Gamma'$ and consequently, the weight $\gamma(z_iP_i'v_i) \notin \Gamma'$. Therefore, $\gamma(z_iP_i'v_iR_1s_1P_i'x_i) \notin \Gamma'$ and the vertex z_i is a breaking vertex of the path $x_iP_i's_1R_1v_iP_i's_2R_2s_3P_i'y_i$. If we consider the linkage $\mathcal{P}'' = (\mathcal{P}' - P_i') \cup x_iP_i's_1R_1v_iP_i's_2R_2s_3P_i'y_i$ and consider $u(z_i)$ and $v(z_i)$ in this linkage, we see that both are contained in the subpath $s_2P_i'v_i$, contradicting our choice of $\mathcal{P}'$ to satisfy E. This final contradiction completes the analysis of the cases and the proof of the lemma. ■

4 Proofs of Lemmas 2.1 and 2.3

We begin with the proof of Lemma 2.1. We will need the following corollary to Theorem 1.3.

Corollary 4.1 *Let G be a graph with $A \subseteq V(G)$ a subset of the vertices. Let $\Sigma \subseteq E(G)$ be a subset of the edges. Then either there exist k vertex disjoint A-paths each containing at least one edge of*

Σ, or there exists a set X of at most $2k-2$ vertices such that every A-path of $G-X$ contains no edge of Σ.

Corollary 4.1 follows by labeling the graph with the group $\mathbb{Z}_2^{E(G)}$ in the natural way so that an A-path has non-zero weight if and only if it contains an edge of Σ.

Proof. *(Lemma 2.1)*

Assume the lemma is false and let Γ be an abelian group, let G be a graph, and let γ be a Γ-labeling of G forming a counter-example to Lemma 2.1 for a subset A of the vertices of G. Furthermore, assume that G and γ are chosen over all such counter-examples so that G has a minimal number of vertices.

First, we establish a minimal amount of connectivity in G.

Claim 4.2 *For any separation (X,Y) of G with $A \subseteq X$, the order of the separation $|X \cap Y|$ is at least two, and if $|X \cap Y| = 2$, then in $G[Y]$, there exist paths R_1 and R_2 linking the two vertices of $X \cap Y$ such that $\gamma(R_1) \neq \gamma(R_2)$.*

Proof. Let (X,Y) be a separation contradicting the claim. If the separation (X,Y) is of order one, then by our choice of (G,γ) to form a counter-example on a minimal number of vertices, we may assume there exists a set Z such that $G[X]-Z$ does not contain any non-zero A-path where $|Z| \leq f_1(k,l)$. But then $G-Z$ also does not contain a non-zero A-path either since no A-path uses a vertex of $Y-X$, contradicting our choice of (G,γ) to be a counter-example. Assume now that the separation (X,Y) is of order exactly two but that every path in $G[Y]$ linking the vertices of $X \cap Y$ has weight $\alpha \in \Gamma$. Let $X \cap Y = \{x_1, x_2\}$ and let G' be the graph $G[X]$ with an edge linking x_1 and x_2 if they are not connected by an edge of G. Consider the group labeled graph (G',γ) where the edge x_1x_2 has weight $\gamma(x_1x_2)=\alpha$. If G' contains either many disjoint non-zero A-paths or a large non-zero A-star, the graph G must as well since at most one composite path can use the edge x_1x_2 and that edge can be replaced in G by a path linking x_1 and x_2 in $G[Y]$ of weight α. Alternatively, if there exists a set Z of size at most $f_1(k,l)$ hitting every non-zero A-path in G', then the set Z will also hit every non-zero A-path in G, contradicting our choice of G to be a counter-example. This completes the proof of the claim. ■

Let Γ' be the subgroup of Γ consisting of 0 and every element of Γ of order two. Let $\mathcal{P}$ be a linkage with components $P_1, \dots, P_n$ such that each non-trivial composite path P_i is either an A-path containing edge e of weight $\gamma(e) \in \Gamma - \Gamma'$ or satisfies $\gamma(P_i) \neq 0$. We define an objective function *value* as follows:

$$value(\mathcal{P}) = 3|\{i : \gamma(P_i) \neq 0\}| + |\{i : \ P_i \text{ is a nontrivial path and } \gamma(P_i) = 0\}|.$$

Claim 4.3 *Let $\mathcal{P}$ be a linkage with components $P_1, \dots, P_n$ such that each non-trivial composite path P_i is either an A-path containing an edge e of weight $\gamma(e) \in \Gamma - \Gamma'$ or satisfies $\gamma(P_i) \neq 0$. Then $value(\mathcal{P}) < 2lk + 3k$.*

Proof. Let $\mathcal{P}$ be a linkage with components $P_1, \dots, P_n$ as in the statement and assume, to reach a contradiction, that $value(\mathcal{P}) \geq 2lk + 3k$. Furthermore, assume $\mathcal{P}$ is chosen over all such linkages to maximize the function *value*. If P_i is a component of $\mathcal{P}$ such that $\gamma(P_i) = 0$, then P_i contains an edge e with $\gamma(e) \in \Gamma - \Gamma'$. It follows that one endpoint of e is a breaking vertex for the path P_i. Consider the linkage $\overline{\mathcal{P}} = \mathcal{P} \cup A$ where we consider each additional vertex of A as a trivial path of length zero. By Lemma 3.1 and Claim 4.2, we may assume that every non-trivial component of $\overline{\mathcal{P}}$ either has non-zero weight or has a breaking vertex that is the attachment of a stable bridge.

Let P_i be a non-trivial component of $\overline{\mathcal{P}}$ with $\gamma(P_i) = 0$. Let B be a stable bridge attaching to the breaking vertex z_i of P_i. We claim that B cannot attach to any other component P of $\overline{\mathcal{P}}$ with $\gamma(P) = 0$. Assume B does attach to such a P at the vertex s. Let the ends of P be x and y (with possibly $x = y$ when P is trivial), and let the ends of P_i be x_i and y_i. There exists a path R contained in B linking s and z_i and otherwise disjoint from $\overline{\mathcal{P}}$. Either $\gamma(xPsRz_iP_ix_i)$ or $\gamma(xPsRz_iP_iy_i)$ must be non-zero since $\gamma(z_iP_ix_i) \neq \gamma(z_iP_iy_i)$. In either case, we contradict our choice of $\overline{\mathcal{P}}$ to maximize *value* as either the linkage $\overline{\mathcal{P}} - \{P_i, P\} \cup xPsRz_iP_ix_i$ or $\overline{\mathcal{P}} - \{P_i, P\} \cup xPsRz_iP_iy_i$ would increase *value*.

Let P_i and P_j be two non-trivial components of $\overline{\mathcal{P}}$ such that $\gamma(P_i) = \gamma(P_j) = 0$, and let B_i and B_j be two stable bridges attaching to a breaking vertex z_i of P_i and a breaking vertex z_j of P_j, respectively. If P is a component of $\overline{\mathcal{P}}$ with $\gamma(P) \neq 0$ containing attachments of both B_i and B_j, then there exists a vertex s of P that is the unique attachment of B_i and B_j on P. Assume otherwise, and that there exists a non-zero path P containing distinct vertices s_i and s_j that are attachments of B_i and B_j, respectively. Let the endpoints of P be x and y and assume that the vertices x, s_1, s_2, y occur on P in that order. Let the endpoints of P_i be x_i and y_i and similarly, the endpoints of P_j be x_j and y_j. As in the previous paragraph, let R_i be a path linking s_i and z_i in B_i and let R_j be defined analogously. Either the path $xPs_iR_iz_iP_ix_i$ or $xPs_iR_iz_iP_iy_i$ must have non-zero weight. Without loss of generality, assume $\gamma(xPs_iR_iz_iP_ix_i) \neq 0$. Similarly, we may assume $\gamma(yPs_jR_jz_jP_jx_j) \neq 0$. We now contradict our choice of $\overline{\mathcal{P}}$ as the linkage $\overline{\mathcal{P}} - \{P_i, P_j, P\} \cup \{xPs_iR_iz_iP_ix_i, yPs_jR_jz_jP_jx_j\}$ violates our choice of $\overline{\mathcal{P}}$ to maximize *value*.

The linkage $\overline{\mathcal{P}}$ contains at most $k-1$ non-trivial components with non-zero weight by our choice of G to be a counter-example. Given that $value(\overline{\mathcal{P}}) \geq 2lk + 3k$, there exist at least $2lk$ non-trivial components in $\overline{\mathcal{P}}$, each with weight zero. It follows that there exists a subset $\mathcal{I}$ of indices of size at least $2l$, a non-zero composite path P of $\overline{\mathcal{P}}$, and a vertex s on P such that the following holds. For all $i \in \mathcal{I}$, P_i is a non-trivial path with $\gamma(P_i) = 0$ containing a breaking vertex z_i such that there exists a bridge attaching to both s and z_i. By our observations in the previous paragraph, we can find internally disjoint paths R_i for all $i \in \mathcal{I}$ such that R_i links z_i and s and R_i is internally disjoint from $\overline{\mathcal{P}}$. We now can construct a non-zero A-star with l composite paths each of the form $x_iP_iz_iR_isR_{i'}z_{i'}P_{i'}x_{i'}$ or $x_iP_iz_iR_isR_{i'}z_{i'}P_{i'}y_{i'}$ for some pair of indices i and i' in $\mathcal{I}$. This contradiction implies that $value(\overline{\mathcal{P}}) < 2kl + 3k$, and consequently, $value(\mathcal{P}) < 2lk + 3k$ as desired by the claim. ∎

An immediate consequence of Claim 4.3 is that there do not exist $2kl + 3k$ disjoint A-paths each containing an edge with weight equal to some element of $\Gamma - \Gamma'$. If we apply Corollary 4.1, we see that there exists a set X of at most $2(2kl + 3k) - 2$ vertices hitting every A-path containing an edge of weight equal to an element of $\Gamma - \Gamma'$. We now fix an arbitrary orientation *dir* of the edges of $G - X$. If the oriented labeling (γ, dir) of $G - X$ contains k disjoint non-zero A-paths, then by Observation 1, $G - X$ would contain k disjoint non-zero A-paths in the unoriented labeling γ. Thus by our choice of X, there exists a set X' of at most $2k - 2$ vertices intersecting all non-zero A-paths in the unoriented labeling γ in the graph $G - X$. We conclude that $X \cup X'$ is a set of at most $2(2kl + 3k) - 2 + (2k - 2)$ vertices intersecting every non-zero A-path in G, contrary to our choice of G as a counter-example. This completes the proof of Lemma 2.1. ∎

We now proceed with the proof of Lemma 2.3.

Proof. *(Lemma 2.3)*

Notice that the statement is vacuously true when $t = 1$. Assume the lemma is false, and choose a counter-example $G = \bigcup_{i=1,\dots,t} \mathcal{S}_i$ for the minimal value of t for which the lemma fails to hold. We first observe that if we delete the vertices $v_1, \dots, v_t$ from G, the resulting graph has maximum degree $2t$. Since we may assume G does not contain k disjoint non-zero A-paths, we see that there exists a set $X \subseteq V(G) - \{v_1, \dots, v_t\}$ with $|X| \leq f_1(k, t+1)$ such that every non-zero A-path of $G - X$ must contain at least one of the nexus vertices $v_1, \dots, v_t$. We discard any composite path containing a vertex in X to construct non-zero A-stars $\mathcal{S}'_1, \dots, \mathcal{S}'_t$ each with at least $n - f_1(k, t+1)$ composite paths. Moreover, every non-zero A-path contained in the subgraph $\bigcup_{i=1,\dots,t} \mathcal{S}'_i$ must contain a nexus vertex.

By our choice of counter-example to minimize t, we see that $\mathcal{S}'_1, \dots, \mathcal{S}'_{t-1}$ contain $t-1$ non-zero A-stars $\mathcal{T}_1, \dots, \mathcal{T}_{t-1}$ such that each $\mathcal{T}_i$ has l composite paths and moreover, for every pair of distinct indices i and j, $\mathcal{T}_i$ and $\mathcal{T}_j$ have no vertex in common. The next claim will complete the proof.

Claim 4.4 *Let $\mathcal{R}_1, \dots, \mathcal{R}_{t'}$ be t' pairwise vertex disjoint non-zero A-stars, each with l composite paths. Let $\mathcal{R}_{t'+1}$ be a non-zero A-star with $8t'l + t' + l$ composite paths. Let v_i be the nexus vertex of $\mathcal{R}_i$ for all $i = 1, \dots, t'+1$, and assume that for all $i \neq j$, $v_i \neq v_j$. Furthermore, assume every non-zero A-path P contained in $\mathcal{R}_1 \cup \dots \cup \mathcal{R}_{t'} \cup \mathcal{R}_{t'+1}$ must contain the nexus vertex v_i of at least one of the A-stars $\mathcal{R}_i$. Then there exist $t'+1$ vertex disjoint non-zero A-stars ${\mathcal{R}'}_1, \dots, {\mathcal{R}'}_{t+1}$ with*

$$V({\mathcal{R}'}_i) \subseteq \bigcup_{j=1,\dots,t'+1} V(\mathcal{R}_j)$$

for all $i = 1, \dots, t'+1$.

Proof. Assume the claim is false, and let $R_1, \dots, R_{t'+1}$ be a counter-example on a minimal number of edges. At most t' composite paths of paths of $R_{t'+1}$ contain a nexus vertex v_j for some $j = 1, \dots, v_{t'}$. By assumption, $R_{t'+1}$ does not have l composite paths that are disjoint from $\bigcup_{j=1,\dots,t'} V(\mathcal{R}_j)$. Also, at most $2t'l$ composite paths of $\mathcal{R}_{t'+1}$ have an endpoint contained in $\mathcal{R}_j$ for some index j. Thus there exist $6t'l$ composite paths $P_1, \dots, P_{6t'l}$ of $\mathcal{R}_{t'+1}$ that satisfy the following conditions:

1. each path P_i contains a vertex in $\mathcal{R}_j$ for some $1 \leq j \leq t'$,
2. no path P_i has an endpoint contained in $\mathcal{R}_j$ for some $j = 1, \dots, t'$, and
3. no path P_i contains a nexus vertex v_j for some $1 \leq j \leq t'$.

Let the ends of P_i be x_i and y_i, with x_i chosen such that the subpath $x_i P_i v_{t'+1}$ intersects some $\mathcal{R}_j$.

The union of the $\mathcal{R}_1, \dots, \mathcal{R}_{t'}$ contains $2t'l$ distinct rays. Thus there exists a ray R with and three distinct indices i such that $x_i P_i v_{t'+1}$ intersects R in a vertex z_i, and moreover, the path $x_i P_i z_i$ is disjoint from the union of $\mathcal{R}_1, \dots, \mathcal{R}_{t'}$ except for the endpoint z_i. Without loss of generality, we may assume that P_1, P_2, and P_3 are three such composite paths of $\mathcal{R}_{t'+1}$ and that the ray R is contained in the composite path Q of $\mathcal{R}_1$. Let x_r be the endpoint of R in A. By the assumption that every non-zero A-path contains a nexus vertex, we see that $\gamma(x_i P_i z_i) = -\gamma(x_r R z_i)$ for $i = 1, 2, 3$. If the subpath $x_i P_i z_i$ has weight $\gamma(x_i P_i z_i)$ equal to 0 or an element of Γ of order two, then $\gamma(x_i P_i z_i) = \gamma(x_r R z_i)$. We conclude that if $\gamma(x_i P_i z_i)$ has order two, then $\mathcal{R}_1 - Q \cup x_i P_i z_i Q, \mathcal{R}_2, \dots, \mathcal{R}_{t'_1}$ is a counter-example to Claim 4.4 on fewer edges.

By our choice of a minimal counter-example, we see that for $i = 1, 2, 3$, the weight $\gamma(x_i P_i z_i)$ must be a non-zero element Γ that is not of order two. It follows that there exist distinct indices

$i, j \in \{1,2,3\}$ such that

$$\gamma(x_i P_i z_i Q z_j P_j x_j) \neq 0$$

contrary to our assumptions since such a path does not contain the nexus vertex of any $\mathcal{R}_k$. To see this, assume the vertices x_r, z_1, z_2, z_3 occur on R in that order. Let $\alpha = \gamma(x_2 P_2 z_2)$. Lest there exist a non-zero path not containing a nexus vertex, both $\gamma(x_3 P_3 z_r R z_2) = \gamma(x_1 P_1 z_1 R z_2) = -\alpha$. But then $\gamma(x_1 P_1 z_1 R z_3 P_3 x_3) = -2\alpha \neq 0$, as desired. ■

Apply Claim 4.4 to the A-stars $\mathcal{T}_1, \ldots, \mathcal{T}_{t-1}, \mathcal{S}'_t$. We then find t pairwise vertex disjoint non-zero A-stars each with l composite paths, proving the lemma. ■

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