Finite connectivity in infinite matroids

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Abstract

We introduce a connectivity function for infinite matroids with properties similar to the connectivity function of a finite matroid, such as submodularity and invariance under duality. As an application we use it to extend Tutte's Linking Theorem to finitary and to cofinitary matroids.

1 Introduction

There seems to be a common misconception[1] that a matroid on an infinite ground set has to sacrifice at least one of the key features of matroids: the existence of bases, or of circuits, or duality. This is not so. As early as 1969 a model of infinite matroids, called B-matroids, was proposed by Higgs [3, 4, 5] that turns out to possess many of the common properties associated with finite matroids. Unfortunately, Higgs' definition and exposition were not very accessible, and although Oxley [6, 7] presented a much simpler definition and made a number of substantial contributions, the usefulness of Higgs' notion remained somewhat obscured. To address this, it is shown in [2] that infinite matroids can be equivalently described by simple and concise sets of independence, basis, circuit and closure axioms, much in the same way as finite matroids. See Section 2 for more details.

In this article, we introduce a connectivity function for infinite matroids and show that it allows one to extend Tutte's Linking Theorem to at least a large class of infinite matroids.

When should we call an infinite matroid k-connected? For $k = 2$ this is easy: a finite matroid is 2-connected, or simply *connected*, if every two elements are contained in a common circuit. This definition can be extended verbatim to infinite matroids. To show that such a definition gives rise to connected components needs some more work, though, as it is non-trivial to show in infinite matroids that containment in a common circuit yields an equivalence relation. We prove this in Section 3.

For larger k, k-connectivity is defined via the connectivity function $\lambda_M(X) = r(X) + r(E - X) - r(M)$, where X is a subset of the ground set of a matroid M and r the rank function. Then, a finite matroid is k-connected unless there exists an ℓ-separation for some $\ell < k$, that is a subset $X \subseteq E$ so that $\lambda_M(X) \leq \ell - 1$ and $|X|, |E - X| \geq \ell$. In an infinite matroid, λ_M makes not much sense as all the involved ranks will normally be infinite. In Section 4 we give an

[1]Compare an earlier (15/03/2010) Wikipedia entry on matroids: "The theory of infinite matroids is much more complicated than that of finite matroids and forms a subject of its own. One of the difficulties is that there are many reasonable and useful definitions, none of which captures all the important aspects of finite matroid theory. For instance, it seems to be hard to have bases, circuits, and duality together in one notion of infinite matroids."

alternative definition of λ_M that defaults to the usual connectivity function for a finite matroid but does extend to infinite matroids. We show, in Section 5, that our connectivity function has some of the expected properties, such as submodularity and invariance under duality.

In a finite matroid, Tutte's Linking Theorem allows the connectivity between two fixed sets to be preserved when taking minors. To formulate this, a refinement of the connectivity function is defined as follows: $\lambda_M(X, Y) = \min\{\lambda_M(U) : X \subseteq U \subseteq E - Y\}$ for any two disjoint $X, Y \subseteq E(M)$.

Theorem 1 (Tutte [9])**.** *Let M be a finite matroid, and let X and Y be two disjoint subsets of $E(M)$. Then there exists a partition (C, D) of $E(M)-(X\cup Y)$ such that $\lambda_{M/C\setminus D}(X, Y) = \lambda_M(X, Y)$.*

As the main result of this article we prove that, based on our connectivity function, Tutte's Linking Theorem extends to a large class of infinite matroids. A matroid is finitary if all its circuits are finite, and it is cofinitary if it is the dual of a finitary matroid.

Theorem 2. *Let M be a finitary or cofinitary matroid, and let X and Y be two disjoint subsets of $E(M)$. Then there exists a partition (C, D) of $E(M)-(X\cup Y)$ such that $\lambda_{M/C\setminus D}(X, Y) = \lambda_M(X, Y)$.*

We conjecture that Theorem 2 holds for arbitrary matroids.

2 Infinite matroids and their properties

Similar to finite matroids, infinite matroids can be defined by a variety of equivalent sets of axioms. Higgs originally defined his B-matroids by giving a set of somewhat technical axioms for the closure operator. This was improved upon by Oxley [6, 7] who gave a far more accessible definition. We follow [2], where simple and consistent sets of independence, basis, circuit, closure and rank axioms are provided.

Let E be some (possibly infinite) set, let $\mathcal{I}$ be a set of subsets of E, and denote by $\mathcal{I}^{\max}$ the inclusion-maximal sets of $\mathcal{I}$. For a set X and an element x, we abbreviate $X \cup \{x\}$ to $X + x$, and we write $X - x$ for $X - \{x\}$. We call $M = (E, \mathcal{I})$ a *matroid* if the following conditions are satisfied:

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\mathcal{I}$ is closed under taking subsets.

(I3) For all $I \in \mathcal{I} - \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$ there is an $x \in I' - I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{ I' \in \mathcal{I} : I \subseteq I' \subseteq X \}$ has a maximal element.

As usual, any set in $\mathcal{I}$ is called *independent*, any subset of E not in $\mathcal{I}$ is *dependent*, and any minimally dependent set is a *circuit.* Any $\subseteq$-maximal set in $\mathcal{I}$ is a *basis.*

Alternatively, we can define matroids in terms of circuit axioms. As for finite matroids we have that (C1) a circuit cannot be empty and that (C2) circuits are incomparable. While the usual circuit exchange axiom does hold in an infinite matroid it turns out to be too weak to define a matroid. Instead, we have the following stronger version:

(C3) Whenever $X \subseteq C \in \mathcal{C}$ and $(C_x : x \in X)$ is a family of elements of $\mathcal{C}$ such that $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C - \left(\bigcup_{x\in X} C_x\right)$ there exists an element $C' \in \mathcal{C}$ such that $z \in C' \subseteq \left(C \cup \bigcup_{x\in X} C_x\right) - X$.

We will need the full strength of (C3) in this paper. The circuit axioms are completed by (CM) which states that the subsets of E that do not contain a circuit satisfy (IM). For more details as well as the basis, closure and rank axioms, see [2].

The main feature of this definition is that even on infinite ground sets matroids have bases, circuits and minors while maintaining duality at the same time. Indeed, most, if not all, standard properties of finite matroids that have a rank-free description carry over to infinite matroids. In particular, every dependent set contains a circuit, every independent set is contained in a basis and duality is defined as one expects, that is, $M^* = (E, \mathcal{I}^*)$ is the dual matroid of a matroid $M = (E, \mathcal{I})$ if $B^* \subseteq E$ is a basis of M^* if and only if $E - B^*$ is a basis of M. The dual of a matroid is always a matroid. As usual, we call independent sets of M^* *coindependent* in M, and similarly we will speak of *codependent* sets, *cocircuits* and *cobases*. Some of the facts above were already proved by Higgs [5] using the language of B-matroids, the other facts are due to Oxley [6]; all of these can be found in a concise manner in [2]. Below we list some further properties that we need repeatedly.

An important subset of matroids are the finitary matroids. Let $\mathcal{I}$ be a set of subsets of E that satisfies (I1) and (I2) as well as the usual augmentation axiom for independent sets, that is, if $I, I' \in \mathcal{I}$ and $|I| < |I'|$ then there exists an $x \in I' - I$ so that $I + x \in \mathcal{I}$. If, in addition, axiom (I4) below holds for $\mathcal{I}$ then we say that $\mathcal{I}$ forms the set of independent sets of a *finitary matroid*.

(I4) $I \subseteq E$ lies in $\mathcal{I}$ if all its finite subsets are contained in $\mathcal{I}$.

Higgs [5] showed that finitary matroids are indeed matroids in our sense. In fact, a matroid is finitary if and only if each of its circuits is finite [2]. A matroid is called *cofinitary* if its dual matroid is finitary.

Finitary matroids occur quite naturally. For instance, the finite-cycle matroid of a graph and any matroid based on linear independence are finitary. The uniform matroids, in which any subset of cardinality at most $k \in \mathbb{N}$ is independent, provide another example. The duals of finitary matroids will normally not be finitary – we will see precisely when this happens in the next section.

Let us continue with a number of useful but elementary properties of (infinite) matroids. Higgs [3] showed that every two bases have the same cardinality if the generalised continuum hypothesis is assumed. We shall only need a weaker statement:

Lemma 3. [2] *If B, B' are bases of a matroid with $|B_1 - B_2| < \infty$ then $|B_1 - B_2| = |B_2 - B_1|$.*

Let $M = (E, \mathcal{I})$ be a matroid, and let $X \subseteq E$. We define the *restriction of M to X*, denoted by $M|X$, as follows: $I \subseteq X$ is independent in $M|X$ if and only if it is independent in M. We write $M \setminus X$ for $M|(E - X)$. We define the *contraction of M to X* by $M^*.X := (M|X)^*$, and we abbreviate $M.(E - X)$ by M/X. Both restrictions and contractions of a matroid are again matroids, see Oxley [7] or [2].

Lemma 4 (Oxley [7]). *Let $X \subseteq E$, and let $B_X \subseteq X$. Then the following are equivalent:*

(i) *B_X is a basis of $M.X$;*

(ii) *there exists a basis B of $M \setminus X$ so that $B_X \cup B$ is a basis of M; and*

(iii) *$B_X \cup B$ is a basis of M for all bases B of $M \setminus X$.*

A proof of the lemma in terms of the independence axioms is contained in [2].

If B is a basis of a matroid M then for any element x outside B there is exactly one circuit contained in $B+x$, the *fundamental circuit of* x, see Oxley [6]. A *fundamental cocircuit* is a fundamental circuit of the dual matroid. We need two more elementary lemmas about circuits.

Lemma 5. *Let M be a matroid and $X \subseteq E(M)$ with $X \neq \emptyset$. If $|D \cap X| \geq 2$ for every cocircuit D of M such that $D \cap X \neq \emptyset$, then X is dependent. If C is a circuit of M, then $|D \cap C| \geq 2$ for every cocircuit D such that $D \cap C \neq \emptyset$.*

Proof. First, assume that $|D \cap X| \geq 2$ for every cocircuit D such that $D \cap X \neq \emptyset$, but contrary to the claim, that X is independent. Then there exists a cobasis B^* contained in $E(M) - X$. If we fix an element $x \in X$ and consider the fundamental cocircuit contained in $B^* + x$, we find a cocircuit intersecting X in exactly one element, namely x, a contradiction. Thus we conclude that X is dependent.

Now consider a circuit C and assume, to reach a contradiction, that there exists a cocircuit D such that $D \cap C = \{x\}$ for some element $x \in C$. The set $D - x$ is coindependent, so there exists a basis B of M disjoint from $D - x$. Now, (IM) yields an independent set I that is $\subseteq$-maximal among all independent sets J with $C - x \subseteq J \subseteq (C - x) \cup B$. By (I3), I is a basis of M, as otherwise there would be an element $b \in B - I$ so that $I + b$ is independent. However, I is disjoint from D, which contradicts that D is codependent. □

Lemma 6. *Let M be a matroid, and let X be any subset of $E(M)$. Then for every circuit C of M/X, there exists a subset $X' \subseteq X$ such that $X' \cup C$ is a circuit of M.*

Proof. Let B_X be a base of $M|X$. The independent sets of M/X are all sets I such that $I \cup B_X$ is independent in M. Since C is a circuit of M/X, it follows that $C \cup B_X$ is dependent and so it contains a circuit C'. By the minimality of circuits, it follows that $C' - X = C$. The set $X' := C' \cap X$ is as desired by the claim. □

3 Connectivity

A finite matroid is connected if and only if every two elements are contained in a common circuit. Clearly, this definition can be extended verbatim to infinite matroids. It is, however, not clear anymore that this definition makes much sense in infinite matroids. Notably, the fact that being in a common circuit is an equivalence relation needs proof. To provide that proof is the main aim of this section.

Let $M = (E, \mathcal{I})$ be a fixed matroid in this section. Define a relation $\sim$ on E by: $x \sim y$ if and only if $x = y$ or if there is a circuit in M that contains x and y. As for finite matroids, we say that M is *connected* if $x \sim y$ for all $x, y \in E$.

Lemma 7. *$\sim$ is an equivalence relation.*

The proof will require two simple facts that we note here.

Lemma 8. *If C is a circuit and $X \subsetneq C$, then $C - X$ is a circuit in M/X.*

Proof. If $C - X$ is not a circuit then there exists a set $C' \subsetneq C - X$ such that C' is a circuit of M/X. Now, Lemma 6 yields a set $X' \subseteq X$ such that $C' \cup X'$ is a circuit of M, and this will be a proper subset of C, a contradiction. □

Lemma 9. *Let $e \in E$ be contained in a circuit of M, and consider $X \subseteq E - e$. Then e is contained in a circuit of M/X.*

Proof. Let e be contained in a circuit C of M, and suppose that e does not lie in any circuit of M/X. Then $\{e\}$ is a cocircuit of M/X, and thus also a cocircuit of M. This, however, contradicts Lemma 5 since the circuit C intersects the cocircuit $\{e\}$ in exactly one element. □

Proof of Lemma 7. Symmetry and reflexivity are immediate. To see transitivity, let e, f, and g in E be given such that e, f lie in a common circuit C_1, and f, g are contained in a circuit C_2. We will find a subset X of the ground set such that M/X contains a circuit containing both e and g. By Lemma 6, this will suffice to prove the claim.

First, we claim that without loss of generality we may assume that

$$E(M) = C_1 \cup C_2 \textit{ and } C_1 \cap C_2 = \{f\}. \tag{1}$$

Indeed, as any circuit in any restriction of M is still a circuit of M, we may delete any element outside $C_1 \cup C_2$. Moreover, we may contract $(C_1 \cap C_2) - f$. Then $(C_1 - C_2) + f$ is a circuit containing e and f, and similarly, $(C_2 - C_1) + f$ is a circuit containing both f and g by Lemma 8. Any circuit C with $e, g \in C$ in $M/((C_1 \cap C_2) - f)$ will extend to a circuit in M, by Lemma 6. Hence, we may assume (1).

Next, we attempt to contract the set $C_2 - \{f, g\}$. If C_1 is a circuit of $M/(C_2 - \{f, g\})$, then we can find a circuit containing both e and g by applying the circuit exchange axiom $(C2)$ to the circuit C_1 and the circuit $\{f, g\}$. Thus we may assume that C_1 is not a circuit, but by Lemma 9, it contains a circuit C_3 containing the element e. If the circuit C_3 also contains the element f, then again by the circuit exchange axiom, we can find a circuit containing both e and g. Therefore, we instead assume that C_3 does not contain the element f. Consequently, there exists a non-empty set $A \subseteq C_2 - \{f, g\}$ such that $C_3 \cup A$ is a circuit of M.

Contract the set $C_2 - (\{f, g\} \cup A)$. We claim that the set $C_3 \cup A$ is a circuit of the contraction. If not, there exist sets $D \subseteq C_3$ and $B \subseteq A$ such that $D \cup B$ is a circuit of $M/(C_2 - (\{f, g\} \cup A))$. Furthermore, $D \cup B \cup X$ is a circuit of M for some set $X \subseteq C_2 - (\{f, g\} \cup A)$. This implies that $D = C_3$, since D contains a circuit of $M/(C_2 - \{f, g\})$. If $A \neq B$, we apply the circuit exchange axiom to the two circuits $C_3 \cup A$ and $C_3 \cup B \cup X$ to find a circuit contained in their union that does not contain the element e. However, the existence of such a circuit is a contradiction. Either it would be contained as a strict subset of C_2, or upon contracting $C_2 - \{f, g\}$ we would have a circuit contained as a strict subset of C_3. This final contradiction shows that $C_3 \cup A$ is a circuit of $M/\left(C_2 - (A \cup \{f, g\})\right)$.

We now consider two circuits in $M/\left(C_2 - (A \cup \{f, g\})\right)$. The first is $C_1' := C_3 \cup A$, which contains e. The second is $C_2' := \{f, g\} \cup A$, the remainder of C_2 after contracting $C_2 - (A \cup \{f, g\})$ (note Lemma 8). We have shown that in attempting to find a circuit containing e and g utilising two circuits C_1 containing e and C_2 containing g, we can restrict our attention to the case when $C_2 - C_1$ consists of exactly two elements. The argument was symmetric, so in fact we may assume that $C_1 - C_2$ also consists of only two elements. In (1) we observed that we may assume that C_1 and C_2 intersect in exactly one element. Thus we have reduced to a matroid on five elements, in which it is easy to find a circuit containing both e and g. □

Let the equivalence classes of the relation $\sim$ be the *connected components* of a matroid M.

As an application of Lemma 7 we shall show that every matroid is the direct sum of its connected components. With a little extra effort this will allow us to re-prove a characterisation of matroids which are both finitary and cofinitary, that had been noted by Las Vergnas [10], and by Bean [1] before.

Let $M_i = (E_i, \mathcal{I}_i)$ be a collection of matroids indexed by a set I, where the ground sets E_i are pairwise disjoint. We define the *direct sum* of the M_i, written $\bigoplus_{i \in I} M_i$, to have ground set consisting of $E := \bigcup_{i \in I} E_i$ and independent sets $\mathcal{I} = \left\{\bigcup_{i \in I} J_i : J_i \in \mathcal{I}_i\right\}$.

As noted by Oxley [7] for finitary matroids, it is easy to check that:

Lemma 10. *The direct sum of matroids M_i for $i \in I$ is a matroid.*

Lemma 11. *Every matroid is the direct sum of the restrictions to its connected components.*

Proof. Let $M = (E, \mathcal{I})$ be a matroid. As $\sim$ is an equivalence relation, the ground set E partitions into connected components E_i, for some index set I. Setting $M_i := M|E_i$, we claim that $\bigoplus_{i \in I} M_i$ and M have the same independent sets.

Clearly, if I is independent in M, then $I \cap E_i$ is independent in M_i for every $i \in I$, which implies that I is independent in $\bigoplus_{i \in I} M_i$. Conversely, consider a set $X \subseteq E$ that is dependent in M. Then, X contains a circuit C, which, in turn, lies in E_j for some $j \in I$. Therefore, $X \cap E_j$ is dependent, implying that X is dependent in $\bigoplus_{i \in I} M_i$ as well. □

We now give the characterisation of matroids that are both finitary and cofinitary.

Theorem 12. *A matroid M is both finitary and cofinitary if and only if there exists an index set I and finite matroids M_i for $i \in I$ such that $M = \bigoplus_{i \in I} M_i$.*

Theorem 12 is a direct consequence of the following lemma, which has previously been proved by Bean [1]. The theorem was first proved by Las Vergnas [10]. Our proof is different from the proofs of Las Vergnas and of Bean.

Lemma 13. *An infinite, connected matroid contains either an infinite circuit or an infinite cocircuit.*

Proof. Assume, to reach a contradiction, that M is a connected matroid with $|E(M)| = \infty$ such that every circuit and every cocircuit of M is finite. Fix an element $e \in E(M)$ and let $C_1, C_2, C_3, \ldots$ be an infinite sequence of distinct circuits each containing e. Let $M' = M|\left(\bigcup_{i=1}^{\infty} C_i\right)$ be the restriction of M to the union of all the circuits C_i. Note that M' contains a countable number of elements by our assumption that every circuit is finite. Let $e_1, e_2, \ldots$ be an enumeration of $E(M')$ such that $e_1 = e$. We now recursively define an infinite set $\mathcal{C}_i$ of circuits and a finite set X_i for $i \geq 1$. Let $\mathcal{C}_1 = \{C_i : i \geq 1\}$ and $X_1 = \{e_1\}$. Assuming $\mathcal{C}_i$ and X_i are defined for $i = 1, 2, \ldots, k$, we define $\mathcal{C}_{k+1}$ as follows. If infinitely many circuits in $\mathcal{C}_k$ contain e_{k+1}, we let $\mathcal{C}_{k+1} = \{C \in \mathcal{C}_k : e_{k+1} \in C\}$, and $X_{k+1} = X_k \cup \{e_{k+1}\}$. Otherwise we set $\mathcal{C}_{k+1} = \{C \in \mathcal{C}_k : e_{k+1} \notin C\}$ and $X_{k+1} = X_k$. Let $X = \bigcup_1^{\infty} X_i$. Note that $\mathcal{C}_k$ is always an infinite set, and that if $e_i \in X_i$, then $e_i \in C$ for all circuits $C \in \mathcal{C}_j$ and i, j with $i < j$.

We claim that the set X is dependent in M' . By Lemma 5, if X is independent then there is a cocircuit D of M' that meets X in exactly one element. As D is finite, we may pick an integer k such that $D \subseteq \{e_1, \ldots, e_k\}$. Choose any $C \in \mathcal{C}_k$. Since $C \cap \{e_1, e_2, e_3, \ldots, e_k\} = X_k$, we see that C also intersects D in exactly one element, a contradiction to Lemma 5. Thus, X is dependent and therefore contains a circuit C'. As M is finitary, C' contains a finite number of elements, and so $C' \subseteq X_\ell$ for some integer ℓ. However, $\mathcal{C}_\ell$ contains an infinite number of circuits, each containing the set X_ℓ. It follows that some circuit strictly contains C', a contradiction. □

Proof of Theorem 12. If we let $\mathcal{C}(M)$ be the set of circuits of the matroid M, an immediate consequence of the definition of the direct sum is that $\mathcal{C}\left(\bigoplus_{i\in I} M_i\right) = \bigcup_{i\in I} \mathcal{C}(M_i)$. Moreover, the dual version of Lemma 5 shows that every cocircuit is completely contained in some M_i. It follows that if $M = \bigoplus_{i\in I} M_i$ where M_i is finite for all $i \in I$, then M is both finitary and cofinitary.

Conversely, let M be a matroid that is both finitary and cofinitary. Let $\{M_i : i \in I\}$ be the set of restrictions of M to its connected components. For every $i \in I$, the matroid M_i is connected, so by our assumptions on M and by Lemma 13, M_i must be a finite matroid. Lemma 11 implies that $M = \bigoplus_{i\in I} M_i$, and the theorem is proved.

□

4 Higher connectivity

Let us recapitulate the definition of k-connectivity in finite matroids and see what we can keep of that in infinite matroids. Let M be a finite matroid on a ground set E. If r_M denotes the rank function then the *connectivity function* λ is defined as

$$\lambda_M(X) := r_M(X) + r_M(E - X) - r_M(E) \text{ for } X \subseteq E. \tag{2}$$

(We note that some authors define a connectivity function λ by $\lambda(X) = k(X)+1$. In dropping the $+1$ we follow Oxley [8].) We call a partition (X, Y) of E a *k-separation* if $\lambda_M(X) \leq k - 1$ and $|X|, |Y| \geq k$. The matroid M is *k-connected* if there exists no ℓ-separation with $\ell < k$.

Of these notions only the connectivity function is obviously useless in an infinite matroid, as the involved ranks will usually be infinite. We shall therefore only redefine λ and leave the other definitions unchanged. For this we have two aims. First, the new λ should coincide with the ordinary connectivity function if the matroid is finite. Second, λ should be consistent with connectivity as defined in the previous section.

Our goal is to find a rank-free formulation of (2). Observe that (2) can be interpreted as the number of elements we need to delete from the union of a basis of $M|X$ and a basis of $M|(E-X)$ in order to obtain a basis of the whole matroid. To show that this number does not depend on the choice of bases is the main purpose of the next lemma.

Let $M = (E, \mathcal{I})$ be a matroid, and and let I, J be two independent sets. We define

$$\mathrm{del}_M(I, J) := \min\{|F| : F \subseteq I \cup J,\ (I \cup J) - F \in \mathcal{I}\},$$

where we set $\mathrm{del}_M(I, J) = \infty$ if there is no such finite set F. Thus, $\mathrm{del}_M(I, J)$ is either a non-negative integer or infinity. If there is no chance of confusion, we will simply write $\mathrm{del}(I, J)$ rather than $\mathrm{del}_M(I, J)$

Lemma 14. *Let $M = (E, \mathcal{I})$ be a matroid, let (X, Y) be a partition of E, and let B_X be a basis of $M|X$ and B_Y a basis of $M|Y$. Then*

(i) $\mathrm{del}(B_X, B_Y) = |F|$ *for any $F \subseteq B_X \cup B_Y$ so that $(B_X \cup B_Y) - F$ is a basis of M.*

(ii) $\mathrm{del}(B_X, B_Y) = |F|$ *for any $F \subseteq B_X$ so that $(B_X - F) \cup B_Y$ is a basis of M.*

(iii) $\mathrm{del}(B_X, B_Y) = \mathrm{del}(B'_X, B'_Y)$ *for every basis B'_X of $M|X$ and basis B'_Y of $M|Y$.*

Proof. Let us first prove that

$$\text{if for } F_1, F_2 \subseteq B_X \cup B_Y, \text{ the set } B_i := (B_X \cup B_Y) - F_i \text{ is a basis of } M \text{ for } i = 1, 2, \text{ then } |F_1| = |F_2|. \tag{3}$$

We may assume that one of $|F_1|$ and $|F_2|$ is finite, say $|F_2|$. Then as $|B_1 - B_2| = |F_2 - F_1| < \infty$, it follows from Lemma 3 that $|F_1 - F_2| = |F_2 - F_1|$, and hence $|F_1| = |F_2|$.

(i) Let $F' \subseteq B_X \cup B_Y$ have minimal cardinality so that $(B_X \cup B_Y) - F'$ is independent. If $|F'| = \infty$ then, evidently, F as in (i) needs to be an infinite set, too. On the other hand, if $|F'| < \infty$ then F' is also $\subseteq$-minimal, and thus $(B_X \cup B_Y) - F'$ is maximally independent in $B_X \cup B_Y$, and hence a basis of M. Now, $|F| = |F'|$ follows with (3).

(ii) Follows directly from (3) and (i).

(iii) Let $F \subseteq B_X$ as in (ii), i.e. $|F| = \mathrm{del}(B_X, B_Y)$. Because of the equivalence of (ii) and (iii) in Lemma 4, we obtain that $(B_X - F) \cup B'_Y$ is a basis of M as well, and it follows that $\mathrm{del}(B_X, B_Y) = |F| = \mathrm{del}(B_X, B'_Y)$. By exchanging the roles of X and Y we get then that $\mathrm{del}(B_X, B'_Y) = \mathrm{del}(B'_X, B'_Y)$, which finishes the proof. □

We now give a rank-free definition of the *connectivity function.* Let X be a subset of $E(M)$ for some matroid M. We pick an arbitrary basis B of $M|X$, and a basis B' of $M \setminus X$ and define $\lambda_M(X) := \mathrm{del}_M(B, B')$. Lemma 14 (iii) ensures that λ is well-defined, i.e. that the value of $\lambda_M(X)$ only depends on X (and M) and not on the choice of the bases. The next two propositions demonstrate that λ extends the connectivity function of a finite matroid and, furthermore, is consistent with connectivity defined in terms of circuits.

Lemma 15. *If M is a finite matroid on ground set E, and if $X \subseteq E$ then*

$$r(X) + r(E - X) - r(E) = \lambda(X).$$

Proof. Let B be a basis of $M|X$, B' a basis of $M \setminus X$, and choose $F \subseteq B \cup B'$ so that $(B \cup B') - F$ is a basis of M. Then

$$\begin{aligned} \lambda(X) &= \mathrm{del}(B, B') = |F| = |B| + |B'| - |(B \cup B') - F| \\ &= r(X) + r(E - X) - r(E). \end{aligned}$$

□

Lemma 16. *A matroid is 2-connected if and only if it is connected.*

Proof. Let $M = (E, \mathcal{I})$ be a matroid. First, assume that there is a 1-separation (X, Y) of M. We need to show that M cannot be connected. Pick $x \in X$ and $y \in Y$ and suppose there is a circuit C containing both x and y. Then $C \cap X$ as well as $C \cap Y$ is independent, and so there are bases $B_X \supseteq C \cap X$ of $M|X$ and $B_Y \supseteq C \cap Y$ of $M|Y$, by (IM). As (X, Y) is a 1-separation, $B_X \cup B_Y$ needs to be a basis. On the other hand, we have $C \subseteq B_X \cup B_Y$, a contradiction.

Conversely, assume M to be 2-connected and pick a $x \in E$. Define X to be the set of all x' so that x' lies in a common circuit with x. If $X = E$ then we are done by Lemma 7. So suppose that $Y := E - X$ is not empty, and choose a basis B_X of $M|X$ and a basis B_Y of $M|Y$. Since there are no 1-separations of M, $B_X \cup B_Y$ is dependent and thus contains a circuit C. But then C must meet X as well as Y, yielding together with Lemma 7 a contradiction to the definition of X. □

To illustrate the definition of λ and since it is relevant for the open problem stated below let us consider an example. Let T_∞ be the ω-regular infinite tree, that is, the tree in which every vertex has countably infinite degree. We call any edge set in T_∞ *independent* if it does not contain a double ray (a 2-way infinite path). The independent sets form a matroid MT_∞ [2]. (It is, in fact, not hard to directly verify the independence axioms.)

What is the connectivity of MT_∞? Since every two edges of T_∞ are contained in a common double ray, we see that M is 2-connected. On the other hand, M contains a 2-separation: deleting an edge e splits the graph T_∞ into two components K_1, K_2. Put $X := E(K_1) + e$ and $Y := E(K_2)$, and pick a basis B_X of $M|X$ and a basis B_Y of $M|Y$. Clearly, neither B_X nor B_Y contains a double ray, while every double ray in $B_X \cup B_Y$ has to use e. Thus, $(B_X \cup B_Y) - e$ is a basis of M, and (X, Y) therefore a 2-separation.

It is easy to construct matroids of connectivity k for arbitrary positive integers k. Moreover, there are matroids that have infinite connectivity, namely the uniform matroid $U_{r,k}$ where $k \simeq r/2$. However, it can be argued that these matroids are simply too small for their high connectivity, and therefore more a fluke of the definition than a true example of an infinitely connected matroid. Such a matroid should certainly have an infinite ground set.

Problem 17. *Find an infinite infinitely connected matroid.*

As for finite matroids the minimal size of a circuit or cocircuit is an upper bound on the connectivity. So, an infinitely connected infinite matroid cannot have finite circuits or cocircuits. In the matroid MT_∞ above all circuits and cocircuits are infinite. Nevertheless, MT_∞ fails to be 3-connected.

5 Properties of the connectivity function

In this section we prove a number of lemmas that will be necessary in extending Tutte's Linking Theorem to finitary matroids. As a by-product we will see that a number of standard properties of the connectivity function of a finite matroid extend to infinite matroids, see specifically Lemmas 18, 19, and 21.

Let us start by showing that connectivity is invariant under duality.

Lemma 18. *For any matroid M and any $X \subseteq E(M)$ it holds that $\lambda_M(X) = \lambda_{M^*}(X)$.*

Proof. Set $Y := E(M) - X$, let B_X be a basis of $M|X$, and B_Y a basis of $M|Y$. By (IM), we can pick $F_X \subseteq B_X$ and $F_Y \subseteq B_Y$ so that $(B_X - F_X) \cup B_Y$ and $B_X \cup (B_Y - F_Y)$ are bases of M.

Then $B_X^* := (X - B_X) \cup F_X$ and $B_Y^* := (Y - B_Y) \cup F_Y$ are bases of $M^*|X$ and $M^*|Y$, respectively. Indeed, $B_X - F_X$ is a basis of $M.X$, by Lemma 4, which implies that $X - (B_X - F_X) = B_X^*$ is a basis of $(M.X)^* = M^*|X$. For B_Y^* we reason in a similar way.

Moreover, since $B_X \cup (B_Y - F_Y)$ is a basis of M we see from

$$(B_X^* - F_X) \cup B_Y^* = (X - B_X) \cup (Y - B_Y) \cup F_Y = E(M) - (B_X \cup (B_Y - F_Y))$$

that $(B_X^* - F_X) \cup B_Y^*$ is a basis of M^*. Therefore

$$\mathrm{del}_M(B_X, B_Y) = |F_X| = \mathrm{del}_{M^*}(B_X^*, B_Y^*),$$

and thus $\lambda_M(X) = \lambda_{M^*}(X)$, as desired. □

Lemma 19. *The connectivity function λ of a matroid M is submodular, that is, for all $X, Y \subseteq E(M)$:*

$$\lambda(X) + \lambda(Y) \geq \lambda(X \cup Y) + \lambda(X \cap Y).$$

Proof. Denote the ground set of M by E. Choose a basis $B_\cap$ of $M|(X \cap Y)$, and a basis $B_{\overline{\cap}}$ of $M \setminus (X \cup Y)$. Pick $F \subseteq B_\cap \cup B_{\overline{\cap}}$ so that $I := (B_\cap \cup B_{\overline{\cap}}) - F$ is a basis of $M|(X \cap Y) \cup (E - (X \cup Y))$. Next, we use (IM) to extend I into $(X-Y) \cup (Y-X)$: let $I_{X-Y} \subseteq X-Y$ and $I_{Y-X} \subseteq Y-X$ so that $I \cup I_{X-Y} \cup I_{Y-X}$ is a basis of M.

We claim that $I_\cup := B_\cap \cup I_{X-Y} \cup I_{Y-X}$ (and by symmetry also $I_{\overline{\cup}} := B_{\overline{\cap}} \cup I_{X-Y} \cup I_{Y-X}$), is independent. Suppose that $I_\cup$ contains a circuit C.

For each $x \in F \cap C$, denote by C_x the (fundamental) circuit in $I \cup \{x\}$. As C meets $I_{X-Y} \cup I_{Y-X}$, we have $C \nsubseteq \bigcup_{x \in F \cap C} C_x$. Thus, (C3) is applicable and yields a circuit $C' \subseteq (C \cup \bigcup_{x \in F \cap C} C_x) - F$. As therefore C' is a subset of the independent set $I \cup I_{X-Y} \cup I_{Y-X}$, we obtain a contradiction.

Since $I_\cup$ is independent and $B_\cap \subseteq I_\cup$ a basis of $M|(X \cap Y)$, we can pick $F_\cup^X \subseteq X - (Y \cup I_\cup)$ and $F_\cup^Y \subseteq Y - (X \cup I_\cup)$ so that $I_\cup \cup F_\cup^X \cup F_\cup^Y$ is a basis of $M|(X \cup Y)$. In a symmetric way, we pick $F_{\overline{\cup}}^X \subseteq X - (Y \cup I_{\overline{\cup}})$ and $F_{\overline{\cup}}^Y \subseteq Y - (X \cup I_{\overline{\cup}})$ so that $I_{\overline{\cup}} \cup F_{\overline{\cup}}^X \cup F_{\overline{\cup}}^Y$ is a basis of $M \setminus (X \cap Y)$.

Let us compute a lower bound for $\lambda(X)$. Both sets $I_X := B_\cap \cup I_{X-Y} \cup F_\cup^X$ and $I_{\overline{X}} := B_{\overline{\cap}} \cup I_{Y-X} \cup F_{\overline{\cup}}^Y$ are independent. As furthermore $I_X \subseteq X$ and $I_{\overline{X}} \subseteq E - X$, we obtain that $\lambda(X) \geq \mathrm{del}(I_X, I_{\overline{X}})$. Since each $x \in F \cup F_\cup^X \cup F_{\overline{\cup}}^Y$ gives rise to a circuit in $I \cup I_{X-Y} \cup I_{Y-X} \cup \{x\}$, we get that $\mathrm{del}(I_X, I_{\overline{X}}) \geq |F| + |F_\cup^X| + |F_{\overline{\cup}}^Y|$. In a similar way we obtain a lower bound for $\lambda(Y)$. Together these result in

$$\lambda(X) + \lambda(Y) \geq 2|F| + |F_\cup^X| + |F_\cup^Y| + |F_{\overline{\cup}}^X| + |F_{\overline{\cup}}^Y|.$$

To conclude the proof we compute upper bounds for $\lambda(X \cap Y)$ and $\lambda(X \cup Y)$. Since $B_\cap$ is a basis of $M|(X \cap Y)$ and $B_{\overline{\cup}} := I_{\overline{\cup}} \cup F_{\overline{\cup}}^X \cup F_{\overline{\cup}}^Y$ is one of $M \setminus (X \cap Y)$, it holds that $\lambda(X \cap Y) = \mathrm{del}(B_\cap, B_{\overline{\cup}})$. Since $I \cup I_{X-Y} \cup I_{Y-X}$ is independent, we get that $\mathrm{del}(B_\cap, B_{\overline{\cup}}) \leq |F| + |F_{\overline{\cup}}^X| + |F_{\overline{\cup}}^Y|$. For $\lambda(X \cup Y)$ the computation is similar, so that we obtain

$$\lambda(X \cap Y) + \lambda(X \cup Y) \leq 2|F| + |F_\cup^X| + |F_\cup^Y| + |F_{\overline{\cup}}^X| + |F_{\overline{\cup}}^Y|,$$

as desired. □

Lemma 20. *In a matroid M let $(X_i)_{i \in I}$ be a family subsets of $E(M)$ with $X_i \supseteq X_j$ if $i \leq j$. Set $X := \bigcap_{i \in I} X_i$. If $\lambda(X_i) \leq k$ for all $i \in I$ then $\lambda(X) \leq k$.*

Proof. Set $Y_i := E(M) - X_i$ for $i \in I$, $Y := \bigcap_{i \in I} Y_i = Y_1$ and $Z := E(M) - (X \cup Y)$. Pick bases B_X of $M|X$ and B_Y of $M|Y$. Choose $I_Z \subseteq Z$ so that $B_Y \cup I_Z$ is a basis of $M|(Y \cup Z)$. Moreover, as $\lambda(X_1) \leq k$ there exists a finite set (of size $\leq k$) $F \subseteq B_Y$ so that $B_X \cup (B_Y - F)$ is a basis of $M|(X \cup Y)$, and a (possibly infinite) set $F_Z \subseteq I_Z$ so that $B_X \cup (B_Y - F) \cup (I_Z - F_Z)$ is a basis of M.

Suppose that $k + 1 \leq \lambda(X) = |F| + |F_Z|$. Then choose $j \in I$ large enough so that $|F| + |F_Z \cap Y_j| \geq k + 1$. Use (IM) to extend the independent subset $B_X \cup (I_Z \cap X_j) - F_Z$ of X_j to a basis B of $M|X_j$. The set $B_Y \cup (I_Z \cap Y_j)$ is independent too, and we may extend it to a basis B' of $M|Y_j$. As $B_X \cup B_Y \cup (I_Z - (F_Z \cap X_j)) \subseteq B \cup B'$ we obtain with

$$\lambda(X_j) = \mathrm{del}(B, B') \geq |F| + |F_Z \cap Y_j| \geq k + 1$$

a contradiction. □

For disjoint sets $X, Y \subseteq E(M)$ define

$$\lambda_M(X, Y) := \min\{\lambda_M(U) : X \subseteq U \subseteq E(M) - Y\}.$$

Again, we may drop the subscript M if no confusion is likely.

Lemma 21. *Let M be a matroid, and let $N = M/C \setminus D$ be a minor of M. Let X and Y be disjoint subsets of $E(N)$. Then $\lambda_N(X, Y) \leq \lambda_M(X, Y)$.*

Proof. Let $U \subseteq E(M)$ be such that $X \subseteq U \subseteq E(M) - Y$ and $\lambda_M(U) = \lambda_M(X, Y)$. First suppose that $N = M \setminus D$ for $D \subseteq E(M) - (X \cup Y)$. Pick a basis B'_U of $M|(U-D)$ and extend it to a basis B_U of $M|U$. Define B'_W and B_W analogously for $W := E(M)-U$. Let $F \subseteq B_U \cup B_W$ be such that $(B_U \cup B_W)-F$ is a basis of M. Since B'_U and B'_W are bases of $M|(U-D) = N|(U-D)$ resp. of $N|(W-D)$, and since clearly $(B'_U \cup B'_W) - (F - D)$ is independent it follows that $\lambda_N(X, Y) \leq \lambda_N(U - D) \leq |F| = \lambda_M(X, Y)$.

Next, assume that $N = M/C$ for some $C \subseteq E(M) - (X \cup Y)$. Then, using Lemma 18 we obtain

$$\lambda_N(X, Y) = \lambda_{(M^* \setminus C)^*}(X, Y) = \lambda_{M^* \setminus C}(X, Y) \leq \lambda_{M^*}(X, Y) = \lambda_M(X, Y).$$

The lemma follows by first contracting C and then deleting D. □

Lemma 22. *In a matroid M let X, Y be two disjoint subsets of $E(M)$, and let $X' \subseteq X$ and $Y' \subseteq Y$ be such that $\lambda(X', Y') = k - 1$. Then $\lambda(X, Y) \geq k$ if and only if there exist $x \in X$ and $y \in Y$ so that $\lambda(X' + x, Y' + y) = k$.*

Proof. Necessity is trivial. To prove sufficiency, assume that there exist no x, y as in the statement. For $x \in X$ denote by $\mathcal{U}_x$ the sets U with $X' + x \subseteq U \subseteq E(M) - Y'$ and $\lambda(U) = k - 1$. By our assumption, $\mathcal{U}_x \neq \emptyset$. By Zorn's Lemma and Lemma 20 there exists an $\subseteq$-minimal element U_x in $\mathcal{U}_x$.

Suppose there is a $y \in Y \cap U_x$. Again by the assumption, we can find a set Z with $X' + x \subseteq Z \subseteq E(M) - (Y' + y)$ and $\lambda(Z) = k - 1$. From Lemma 19 it follows that $\lambda(U_x \cap Z) = k - 1$, and thus that $U_x \cap Z$ is an element of $\mathcal{U}_x$. As $y \notin U_x \cap Z$ it is strictly smaller than U_x and therefore a contradiction to the minimality of U_x. Hence, U_x is disjoint from Y.

Next, let $\mathcal{W}$ be the set of sets W with $Y \subseteq W \subseteq E(M) - X'$ and $\lambda(W) = k - 1$. As $E(M) - U_x \in \mathcal{W}$ for every $x \in X$, $\mathcal{W}$ is non-empty and we can apply Zorn's Lemma and Lemma 20 in order to find an $\subseteq$-minimal element W' in $\mathcal{W}$. Suppose that there is a $x \in X \cap W'$. But then Lemma 19 shows that $W' \cap (E(M) - U_x) \in \mathcal{W}$, a contradiction to the minimality of W'.

In conclusion, we have found that $Y \subseteq W \subseteq E(M) - X$ and $\lambda(W) = k - 1$, which implies $\lambda(X, Y) \leq k - 1$. This contradiction proves the claim. □

6 The Linking Theorem

In this section we prove our main theorem, Tutte's Linking Theorem for finitary (and cofinitary) matroids, which we restate here:

Theorem 2. *Let M be a finitary or cofinitary matroid, and let X and Y be two disjoint subsets of $E(M)$. Then there exists a partition (C, D) of $E(M) - (X \cup Y)$ such that $\lambda_{M/C \setminus D}(X, Y) = \lambda_M(X, Y)$.*

A fact that is related to Tutte's Linking Theorem, but quite a bit simpler to prove, is that for every element e of a finite 2-connected matroid M, one of M/e or $M - e$ is still 2-connected. This fact extends to infinite matroids in a straightforward manner. Yet, in an infinite matroid it is seldom necessary to only delete or contract a single element or even a finite set. Rather, to be useful we would need that

for any set $F \subseteq E(M)$ of a 2-connected matroid $M = (E, \mathcal{I})$ there always exists a partition (A, B) of F so that $M/A \setminus B$ is still 2-connected. (4)

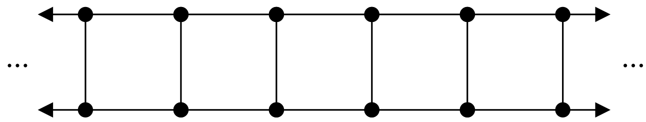

Figure 1: The double ladder

Unfortunately, such a partition of F does not need to exist. Indeed, consider the finite-cycle matroid M_{FC} obtained from the double ladder (see Figure 1), i.e. the matroid on the edge set of the double ladder in which an edge set is independent if and only if it does not contain a finite cycle. If F is the set of rungs then we cannot contract any element in F without destroying 2-connectivity, but if we delete all rungs we are left with two disjoint double rays.

In view of the failure of (4) in infinite matroids, even in finitary matroids like the example, it appears somewhat striking that Tutte's Linking Theorem does extend to, at least, finitary matroids.

Before we can finally prove Theorem 2 we need one more definition and one lemma that will be essential when $\lambda(X,Y) < \infty$. Let $M' = M/C \setminus D$ be a minor of a matroid M, for some disjoint sets $C, D \subseteq E(M)$. We say a k-separation (X', Y') of M' *extends* to a k-separation of M if there exists a k-separation (X, Y) of M such that $X' \subseteq X$ and $Y' \subseteq Y$. The k-separation (X', Y') is *exact* if $\lambda(X', Y') = k - 1$.

Lemma 23. *Let M be a matroid and let $X \cup Y \subseteq E(M)$ be disjoint subsets of $E(M)$. Let (X, Y) be an exact k-separation of $M|(X \cup Y)$ that does not extend to a k-separation of M. Then there exist circuits C_1 and C_2 of M such that (X, Y) does not extend to a k-separation of $M|(X \cup Y \cup C_1 \cup C_2)$.*

Proof. We define

$$Comp_X := \{D : \ D \text{ a component of } M/X \text{ such that } D \cap Y = \emptyset\}$$

to be the set of connected components of M/X that do not contain an element of Y. Symmetrically, we define $Comp_Y$ to be the components of M/Y that do not contain an element of X.

We claim that

$$\textit{if } A \in Comp_X \textit{ and } B \in Comp_Y \textit{ such that } A \cap B \neq \emptyset, \textit{ then } A = B. \tag{5}$$

Assume the claim to be false and let A and B be a counterexample. Without loss of generality, we may assume there exists an element $x \in A \cap B$ and an element $y \in B - A$. By definition (and Lemma 6), there exists a circuit C_Y of M such that $C_Y - Y$ is a circuit of M/Y containing x and y. Now consider the circuit C_Y in the matroid M/X. By Lemma 9, the dependent set $C_Y - X$ (in fact, C_Y is disjoint from X but we will not need this) contains a circuit of M/X that contains x but not y as y and x lie in distinct components of M/X. It follows that there exists a circuit C_X in M such that $x \in C_X - X \subseteq C_Y - y$. By the finite circuit exchange axiom or (C3), there exists a circuit $C \subseteq C_X \cup C_Y$ of M containing y but not x. Hence, there is a circuit $D \subseteq C - Y$ in M/Y with $y \in D$ and $x \notin D$. Since $y \in B$, D cannot meet X, which implies $D \subseteq C - (X \cup Y) \subseteq C_Y - Y$. As $x \notin D$, the circuit D in M/Y is a strict subset of the circuit $C_Y - Y$, a contradiction. This proves the claim.

Note that it is certainly possible that a set A lies in both $Comp_X$ and $Comp_Y$. In a slight abuse of notation, we let $E(Comp_X) = \bigcup_{A \in Comp_X} A$, and similarly define $E(Comp_Y)$.

Next, let us prove that

$$\text{If } E(Comp_X) \cup E(Comp_Y) \cup X \cup Y = E(M) \text{, then the separation } (X,Y) \text{ extends to a } k\text{-separation of } M. \tag{6}$$

Indeed, consider the partition (L,R) for $L := X \cup E(Comp_X)$ and $R := E(M) - L \supseteq Y$. We claim that (L,R) is a k-separation of M. Pick bases B_X and B_Y of $M|X$ resp. of $M|Y$, use (IM) to extend B_X to a basis B_L of $M|L$, and let B_R be a basis of $M|R$ containing B_Y.

Consider a circuit $C \subseteq B_L \cup B_R$ in M, and suppose C to contain an element $x \in B_L - X$. The set $C - X$ contains a circuit C' in M/X containing x. Since $(C \cap B_L) - X$ is independent in M/X, the circuit C' must contain an element $y \in B_R$. This implies that x and y are in the same component of M/X, and consequently, $y \in E(Comp_X)$. This contradicts the definition of the partition, implying that no such circuit C and element x exist. A similar argument implies that $B_L \cup B_R$ does not contain any circuit containing an element of $B_R - Y$ by considering M/Y. We conclude that every circuit contained in $B_L \cup B_R$ must lie in $B_X \cup B_Y$. As $\lambda(X,Y) = k-1$, there exists a set of $k-1$ elements intersecting every circuit contained in $B_X \cup B_Y$, and thus in $B_L \cup B_R$, which implies that (L,R) forms a k-separation. This completes the proof of (6).

Before finishing the proof of the lemma, we need one further claim.

$$\text{Let } C \text{ be a circuit of } M \text{ such that } C - (X \cup Y) \text{ is a circuit of } M/(X \cup Y)\text{. Then the only } k\text{-separations of } M|(X \cup Y \cup C) \text{ that extend } (X,Y) \text{ are } (X \cup (C - Y), Y) \text{ and } (X, Y \cup (C - X)). \tag{7}$$

Assume that (X',Y') is a k-separation that extends (X,Y) in the matroid $M|(X \cup Y \cup C)$. Let $C' := C - (X \cup Y)$. Assume that (X',Y') induces a proper partition of C', i.e. that $C' \cap X' \neq \emptyset$ and $C' \cap Y' \neq \emptyset$. Picking bases B_X of $M|X$ and B_Y of $M|Y$ we observe that $B_X \cup (C' - Y')$ and $B_Y \cup (C' - X')$ form bases of $M|X'$ and $M|Y'$ respectively. Assume there exists a set F of $k-1$ elements intersecting every circuit contained in $B_X \cup B_Y \cup C$. By our assumption that (X,Y) is an exact k-separation, we see that $F \subseteq B_X \cup B_Y$. However, C' is a circuit of $M/(X \cup Y)$, or, equivalently, C' is a circuit of $M/((B_X \cup B_Y) - F)$. It follows that there exists a circuit contained in $C' \cup B_X \cup B_Y$ avoiding the set F, a contradiction. This completes the proof of (7).

Since, by assumption, (X,Y) does not extend to a k-separation of M it follows from (6) that there is an $e \notin E(Comp_X) \cup E(Comp_Y)$. Then there exists a circuit C_1 of M containing e with $C_1 \cap Y \neq \emptyset$ such that the following hold:

- $C_1 - X$ is a circuit of M/X, and
- $C_1 - (X \cup Y)$ is a circuit of $M/(X \cup Y)$.

To see that such a circuit C_1 exists, recall first that $e \notin E(Comp_X)$ implies that there is a circuit C_X in M/X containing e so that $C_X \cap Y \neq \emptyset$. Then $C_X - Y$ contains a circuit C' in $M/(X \cup Y)$ with $e \in C'$ (see Lemma 9). For suitable $A_X \subseteq X$ and $A_Y \subseteq Y$ it therefore holds, by Lemma 6, that $C' \cup A_X \cup A_Y$ is a circuit of M. If $A_Y = \emptyset$ then C' would be a dependent set of M/X strictly contained in the circuit C_X. Thus, $A_Y \neq \emptyset$ and $C_1 := A_X \cup A_Y \cup C'$ has the desired properties. Symmetrically, there exists a circuit C_2 containing e and intersecting X in at least one element such that $C_2 - Y$ is a circuit of M/Y and $C_2 - (X \cup Y)$ is a circuit of $M/(X \cup Y)$.

Let us now see that the circuits C_1 and C_2 are as required in the statement of the lemma. To reach a contradiction, suppose that (X, Y) extends to a k-separation (X', Y') of $M|(X \cup Y \cup C_1 \cup C_2)$. By symmetry, we may assume that $e \in X'$. By (7), we see that $C_1 - (X \cup Y) \subseteq X'$ and $C_2 - (X \cup Y) \subseteq X'$ as well. Pick a basis B_X of $M|X$, and a basis B_Y of $M|Y$. From $C_1 \cap Y \neq \emptyset$ it follows that $C_1 - Y$ is independent in M/X. Thus, $B_X \cup (C_1 - Y)$ is independent and we can extend it with (IM) to a basis $B_{X'}$ of $M|X'$. The set B_Y forms a basis of $M|Y'$ as $Y' = Y$. Choose $F \subseteq B_{X'} \cup B_Y$ so that $(B_{X'} \cup B_Y) - F$ is independent. As (X, Y) is an exact k-separation and (X', Y) thus too, it follows that $|F| = k - 1$. As $\lambda_{M|X \cup Y}(X, Y) = k - 1$, we see $F \subseteq B_X \cup B_Y$. However, the set $C_1 - (X \cup Y)$ is dependent in $M/(X \cup Y)$ and thus in $M/((B_X \cup B_Y) - F)$. Hence, there is a set $S \subseteq (B_X \cup B_Y) - F$ so that $(C_1 - (X \cup Y)) \cup S \subseteq B_{X'} \cup B_Y - F$ is dependent in M, contradicting our choice of F. This contradiction implies that the separation (X, Y) does not extend to a k-separation of $M|(X \cup Y \cup C_1 \cup C_2)$, which concludes the proof of the lemma. $\square$

We now proceed with the proof of the Linking Theorem for finitary matroids.

Proof of Theorem 2. By Lemma 18, $\lambda_M(Z) = \lambda_{M^*}(Z)$ for any $Z \subseteq E(M)$, which means that we may assume M to be finitary. Recall that this implies that every circuit of M is finite. We will consider the cases when $\lambda_M(X, Y) = \infty$ and when $\lambda_M(X, Y) < \infty$ separately.

Assume that $\lambda_M(X, Y) = \infty$. We inductively define a series of disjoint circuits $C_1, C_2, \ldots$ in different minors of M as follows. Starting with C_1 to be chosen as a circuit in M intersecting both X and Y, assume $C_1, \ldots, C_t$ to be defined for $t \geq 1$. Note that as $Z := \bigcup_{i=1}^{t} C_i$ is finite, we still have $\lambda_{M/Z}(X - Z, Y - Z) = \infty$. Thus, there exists a circuit C_{t+1} in M/Z that meets both both $X - Z$ and $Y - Z$. Having finished this construction, we let $C_X = (\bigcup_{i=1}^{\infty} C_i) \cap X$, $C_Y = (\bigcup_{i=1}^{\infty} C_i) \cap Y$, and $C = (\bigcup_{i=1}^{\infty} C_i) - (X \cup Y)$. Set $D = E(M) - (X \cup Y \cup C)$.

In order to show $\lambda_{M/C \setminus D}(X, Y) = \infty$ observe first that C_X (and symmetrically, C_Y) is an independent set in M/C. If not then there exists a circuit $A \subseteq C_X \cup C$. Given that M is finitary and A thus finite, there exists a minimal index t such that $A \subseteq \bigcup_{i=1}^{t} C_i$. It follows that $A - (\bigcup_{i=1}^{t-1} C_i)$ is dependent in $M/(\bigcup_{i=1}^{t-1} C_i)$. Since A is disjoint from Y but $C_t \cap Y \neq \emptyset$, the dependent set $A - (\bigcup_{i=1}^{t-1} C_i)$ is also a strict subset of the circuit C_t, a contradiction.

Let B_X be a basis of X containing C_X, and let B_Y be a basis of Y containing C_Y in $M/C \setminus D$. Assume there exists a finite set F such that $(B_X \cup B_Y) - F$ is a basis of $M/C \setminus D$. Then for all $f \in F$, there exists a (fundamental) circuit $A_f \subseteq B_X \cup B_Y$ of $M/C \setminus D$ with $A_f \cap F = \{f\}$. Since the circuits A_f are finite and the C_i pairwise disjoint, we may choose t large enough so that $C_t \cap A_f = \emptyset$ for all $f \in F$. Lemma 9 ensures the existence of a circuit $K \subseteq C_X \cup C_Y$ in $M/C \setminus D$ with $K - \bigcup_{f \in F} A_f \neq \emptyset$. By the finite circuit exchange axiom or (C3), there exists a circuit contained in $(K \cup \bigcup_{f \in F} A_f) - F \subseteq (B_X \cup B_Y) - F$, a contradiction. It follows that $\lambda_{M/C \setminus D}(X, Y) = \infty$, as claimed.

We now consider the case when $\lambda_M(X,Y) = k < \infty$. By repeatedly applying Lemma 22, there exists a set $X' \subseteq X$ and $Y' \subseteq Y$ such that $\lambda_M(X',Y') = k$ and $|X'| = |Y'| = k$. (Observe that $\lambda(X',Y') = k$ implies $|X'|, |Y'| \geq k$.) We shall find a partition (C', D') of $E(M) - (X' \cup Y')$ such that $\lambda_{M/C' \backslash D'}(X',Y') = k$. Then, Lemma 21 implies that setting $C = C' - (X \cup Y)$ and $D = D' - (X \cup Y)$ results in $\lambda_{M/C \backslash D}(X,Y) = k$ as desired.

In order to find such C' and D' we will inductively define for $t \leq k$ finite sets $Z_t \subseteq E(M)$ with $Z_{t-1} \subseteq Z_t$ such that in the restriction $M|Z_t$ it holds that $\lambda_{M|Z_t}(X',Y') \geq t$. For $t = 1$, pick a circuit A intersecting both X' and Y', and let $Z_1 = X' \cup Y' \cup A$. As M is finitary Z_1 is finite, and its choice ensures $\lambda_{M|Z_1}(X',Y') \geq 1$.

Assume that for $t < k$ we have defined Z_{t-1}, and and observe that as Z_{t-1} is finite, there are only finitely many t-separations in $M|Z_{t-1}$ separating X' and Y', all of which are exact. By applying Lemma 23 to each of those, we deduce that there exists a finite set of circuits $A_1, A_2, \ldots, A_l$ such that for $Z_t := Z_{t-1} \cup A_1 \cup \cdots \cup A_l$ we get $\lambda_{M|Z_t}(X',Y') \geq t$.

To conclude, note that the matroid $M|Z_k$ is finite and that $\lambda_{M|Z_k}(X',Y') = k$. By Theorem 1, there exists a partition $(C', \tilde{D})$ of $Z_k - (X' \cup Y')$ such that $\lambda_{(M|Z_k)/C' \backslash \tilde{D}}(X',Y') = k$. Consequently, we obtain $\lambda_{M/C' \backslash D'}(X',Y') = k$ for $D' := \tilde{D} \cup (E(M) - Z_k)$, which completes the proof of the theorem. □

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