A Weaker Version of Lovász’ Path Removal Conjecture

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Abstract

We prove there exists a function $f(k)$ such that for every $f(k)$-connected graph $G$ and every edge $e \in E(G)$, there exists an induced cycle $C$ containing $e$ such that $G - E(C)$ is $k$-connected. This proves a weakening of a conjecture of Lovász due to Kriesell.

Key Words: graph connectivity, removable paths, non-separating cycles

1 Introduction

The following conjecture is due to Lovász (see [14]):

Conjecture 1.1 There exists a function $f = f(k)$ such that the following holds. For every $f(k)$-connected graph $G$ and two vertices $s$ and $t$ in $G$, there exists a path $P$ with endpoints $s$ and $t$ such that $G - V(P)$ is $k$-connected.

Conjecture 1.1 can alternately be phrased as following: there exists a function $f(k)$ such that for every $f(k)$-connected graph $G$ and every edge $e$ of $G$, there exists a cycle $C$ containing $e$ such that $G - V(C)$ is $k$-connected. Lovász also conjectured [9] that every $(k+3)$-connected graph contains a cycle $C$ such that $G - V(C)$ is $k$-connected. This was proven by Thomassen [13].

Conjecture 1.1 is known to be true in several small cases. In the case $k = 1$, a path $P$ connecting two vertices $s$ and $t$ such that $G - V(P)$ is connected is called a non-separating path. It follows from a theorem of Tutte that any 3-connected graph contains a non-separating path connecting any two vertices, and consequently, $f(1) = 3$. When $k = 2$, it was independently shown by Chen, Gould, and Yu [1] and Kriesell [6] that $f(2) = 5$. In [1], the authors also show that in a $(22k+2)$-connected graph, there exist $k$ internally disjoint non-separating paths connecting any pair of vertices. In [5], Kawarabayashi, Lee, and Yu obtain a complete structural characterization of which 4-connected graphs do not have a path linking two given vertices whose deletion leaves the graph 2-connected.

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In a variant of the problem, one can attempt to delete the edges of the path instead of deleting all the vertices. Mader proved [11] that every \( k \)-connected graph with minimum degree \( k + 2 \) contains a cycle \( C \) such that deleting the edges of \( C \) leaves the graph \( k \)-connected. Jackson independently proved the same result when \( k = 2 \) in [4]. As a corollary to a stronger result, Lemos and Oxley have shown [8] that in a 4-connected graph \( G \), for any edge \( e \) there exists a cycle \( C \) containing \( e \) such that \( G - E(C) \) is 2-connected.

Kriesell has postulated the following natural weakening of Conjecture 1.1

**Conjecture 1.2 (Kriesell, [7])** There exists a function \( f(k) \) such that for every \( f(k) \)-connected graph \( G \) and any two vertices \( s \) and \( t \) of \( G \), there exists an induced path \( P \) with ends \( s \) and \( t \) such that \( G - E(P) \) is \( k \)-connected.

We answer this question in the affirmative with the following theorem.

**Theorem 1.3** There exists a function \( f(k) = O(k^4) \) such that the following holds: for any two vertices \( s \) and \( t \) of an \( f(k) \)-connected graph \( G \), there exists an induced \( s - t \) path \( P \) such that \( G - E(P) \) is \( k \)-connected.

**Corollary 1.4** For every \( (f(k) + 1) \)-connected graph \( G \) and for every edge \( e \) of \( G \), there exists an induced cycle \( C \) containing \( e \) such that \( G - E(C) \) is \( k \)-connected.

In the proof of Theorem 1.3, we will at several points need to force the existence of highly connected subgraphs using the fact that our graph will have large minimum degree. A theorem of Mader implies the following.

**Theorem 1.5 (Mader, [10])** Every graph of minimum degree \( 4k \) contains a \( k \)-connected subgraph.

In addition to simply requiring a highly connected subgraph, we will require the subgraph have small boundary. The **boundary** of a subgraph \( H \) of a graph \( G \), denoted \( \partial_G(H) \), is the set of vertices in \( V(H) \) that have a neighbor in \( V(G) - V(H) \). We use the following related result of Thomassen. By strengthening the minimum degree condition in Theorem 1.5, we can find a highly connected subgraph that further has a small boundary.

**Theorem 1.6 (Thomassen, [15])** Let \( k \) be any natural number, and let \( G \) be any graph of minimum degree \( > 4k^2 \). Then \( G \) contains a \( k \)-connected subgraph with more than \( 4k^2 \) vertices whose boundary has at most \( 2k^2 \) vertices.

Given a path \( P \) in a graph, and two vertices \( x \) and \( y \) on \( P \), we denote by \( xPy \) the subpath of \( P \) starting at vertex \( x \) and ending at \( y \). A **separation** of a graph \( G \) is a pair \( (A, B) \) of subsets of vertices of \( G \) such that \( A \cup B = V(G) \), and for every edge \( e = uv \) of \( G \), either both \( u \) and \( v \) are contained in \( A \) or both are contained in \( B \). The **order** of a separation \( (A, B) \) is \( |A \cap B| \). Where not otherwise stated, we follow the notation of [2].

We will need the following results on systems of disjoint paths with pre-specified endpoints.

**Definition** A **linkage** is a graph where every connected component is a path.

A **linkage problem** in a graph \( G \) is a set of pairs of vertices in \( G \). We will typically write the linkage problem \( \mathcal{L} \) as follows:

\[
\mathcal{L} = \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}.
\]
A solution to the linkage problem \( L = \{ \{s_1, t_1\}, \ldots, \{s_k, t_k\} \} \) is a set of pair-wise internally disjoint paths \( P_1, \ldots, P_k \) such that the ends of \( P_i \) are \( s_i \) and \( t_i \), and furthermore, if \( x \in V(P_1) \cap V(P_j) \) for some distinct indices \( i \) and \( j \), then \( x = s_i \) or \( x = t_i \). A graph \( G \) is strongly \( k \)-linked if every linkage problem \( L = \{ \{s_1, t_1\}, \ldots, \{s_k, t_k\} \} \) consisting of \( k \) pairs in \( G \) has a solution. The graph \( G \) is \( k \)-linked if every linkage problem with \( k \) pair-wise disjoint pairs of vertices has a solution. We utilize the following theorem:

**Theorem 1.7 ([12])** Every \( 10k \)-connected graph is \( k \)-linked.

Any \( k \)-linked graph on at least \( 2k \) vertices is strongly \( k \)-linked. Thus the following statement follows trivially from Theorem 1.7.

**Corollary 1.8** Every \( 10k \)-connected graph is strongly \( k \)-linked.

## 2 Proof of Theorem 1.3

We prove the theorem with the function \( f(k) = 1600k^4 + k + 2 \). Let \( S \) be a \( 2k \)-connected subgraph of \( G \) such that \( G - E(S) \) contains an induced \( s-t \) path. To see that such a subgraph \( S \) exists, consider an \( s-t \) path \( P_0 \) of minimum length. We note that \( P_0 \) is an induced path, and, further, that \( G - E(P_0) \) has minimum degree \( f(k) - 3 > 8k \). By Theorem 1.5, \( G - E(P_0) \) contains the desired \( 2k \)-connected subgraph \( S \).

Our goal in the proof of Theorem 1.3 will be to pick an \( s-t \) path \( P \) which uses no edges of \( S \) and has the following property. For every vertex \( x \) of \( G \), in the graph \( G - E(P) \) the vertex \( x \) has \( k \) internally disjoint paths to distinct vertices in \( S \). This will suffice to show that \( G - E(P) \) is \( k \)-connected. To find such a path, we pick \( P \) to maximize the number of vertices with \( k \) paths to \( S \), and subject to that, to maximize the number of vertices with \( k-1 \) paths to \( S \), and so on. This leads to the following definition. For any induced \( s-t \) path \( P \) such that \( E(P) \) is disjoint from \( E(S) \), we define the set:

\[
S_k = S_k(P) = \{ v \mid \exists k \text{ internally disjoint paths in } G - E(P) \text{ from } v \text{ to } V(S) \text{ with distinct ends in } V(S) \}.
\]

For \( i \) between 0 and \( k - 1 \) we define sets \( S_i \) where a vertex \( v \) is in \( S_i \) if \( v \) is joined to \( V(S) \) by \( i \) paths in \( G - E(P) \) disjoint except at \( v \) and not \( i + 1 \) such paths.

We choose an induced \( s-t \) path \( P \) disjoint from \( E(S) \) so as to lexicographically maximize

\[
(S_k, S_{k-1}, \ldots, S_0).
\]

It now suffices to show that for this \( P \), \( |S_k| = |V(G)| \). We let \( \text{min} = \min \{ i \mid S_i \neq \emptyset \} \). We will show that if \( \text{min} < k \), there exists an induced path \( P^* \) which avoids \( E(S) \) and satisfies the following properties:

(a) for all \( v \) in \( S_j(P), j > \text{min}, v \in S_{j^*}(P^*) \) for some \( j^* \geq j \),

(b) there exists a \( v \) in \( S_{\text{min}} \) which is in \( S_{j^*}(P^*) \) for some \( j^* > \text{min} \).
This contradicts our choice of \( P \).

To find \( P^* \), observe that there exists a separation \((A, B)\) of \( G-E(P)\) of order \( \min \) with \( V(S) \subseteq A \) and \( v \in B - A \). Assume we have chosen such a separation to minimize \( |A| \). Let \( X \) denote the set \( A \cap B \). It follows from our choice of \( \min \) that every vertex of \( B - A \) is contained in \( S_{\min} \).

Consider the subgraph of \( G \) induced by \( B - A \). We note that \( G[B-A] \) has minimum degree at least \( f(k) - k - 2 = 1600k^4 \). By Theorem 1.6, there exists a \( 20k^2 \)-connected subgraph \( F \) in \( G[B-A] \) of size at least \( 1600k^4 \) which has a boundary of size at most \( 800k^4 \).

By our choice of \( \min \), there exist \( |X| \) disjoint paths from \( X \) to \( F \) in the graph \( G-E(P) \) restricted to the set \( B \). We choose \( |X| \) such paths internally disjoint from \( F \). Let \( X' \) be the endpoints of the paths in \( F \). Let \( L_1 \) be the linkage problem \( \{(x, y)| x, y \in X', x \neq y\} \) consisting of every pair of vertices of \( X' \).

For every vertex \( x \in X, x \in S_t \) for some value of \( t = t(x) \). There exist paths \( Q^*_1, \ldots, Q^*_{t(x)} \) in \( G-E(P) \) disjoint except for the vertex \( x \) each having one endpoint in \( S \) and the other endpoint equal to \( x \). Let \( Q \) be a path in \( G \) with endpoints \( u \) and \( v \). A vertex \( x \in V(F) \cap V(Q) \) is \( Q\text{-extremal} \) if either \( uQx \) or \( xQv \) contains no vertex of \( V(F) \) other than the vertex \( x \). We let \( Q \) be the set of paths \( \{Q^*_i | x \in X, 1 \leq i \leq t(x)\} \). Note, two distinct \( Q_1, Q_2 \in Q \) are not necessarily disjoint. A vertex \( x \in V(F) \) is \( Q\text{-extremal} \) if there exists a path \( Q \in Q \) such that \( x \) is \( Q\text{-extremal} \). Let \( Y' \) be the set of \( Q\text{-extremal} \) vertices in \( V(F) \), and let \( L_2 \) be the natural linkage problem induced by \( Q \):

\[
L_2 = \{(x, y)| x, y \in Y' \text{ and } \exists Q \in Q \text{ such that } x \text{ and } y \text{ are } Q\text{-extremal}\}
\]

Observe that while a vertex in \( X \) may have many neighbors in \( V(F) - \partial_{G[B-A]}(F) \), the only edges of \( G \) with one end in \( A - B \) and the other end in \( V(F) - \partial_{G[B-A]}(F) \) are contained in \( P \). It follows that either \( X' \) or \( Y' \) may contain vertices of \( V(F) - \partial_{G[B-A]}(F) \). See Figure 1.

![Figure 1: An example of the separation (A, B) with the subgraphs S and F and possible sets X' and Y'.](image)

Recall that the size of the boundary of \( F \) is at most \( 800k^4 \) in \( G[B-A] \). It follows from the connectivity of \( G \) that there exists a matching of size three from \( V(F) - X' - Y' - \partial_{G[B-A]}(F) \) to \( A - X \) using only edges of \( P \). Let \( aa', bb' \) and \( cc' \) be three edges forming such a matching where the vertices \( a, b, \) and \( c \) lay in \( V(F) - X' - Y' - \partial_{G[B-A]}(F) \). By our choice of \((A, B)\) to minimize \( |A| \), there exist \( |X| + 1 \) disjoint paths from \( X \cup \{a'\} \) to \( V(S) \) in \( G - E(P) \) (and similarly for \( X \cup \{b'\} \) and \( X \cup \{c'\} \)).
By Theorem 1.7, the graph $F$ is strongly $2k^2$-linked. Fix vertices $s^*$ and $s'$ as follows. Let $s^*$ be a vertex in $V(F) - X' - Y'$ such that $s^*$ has a neighbor $s'$ on $P$ in $G$ and furthermore, assume that $s^*$ and $s'$ are chosen so that $s'$ is as close to $s$ on $P$ as possible. Similarly, we define $t^*$ and $t'$ such that $t^*$ is a vertex of $V(F) - X' - Y'$ with a neighbor $t'$ as close to $t$ as possible. The vertices $s^*$ and $t^*$ are well defined since $a$, $b$, and $c$ all have a neighbor on $P$ in $G$. Without loss of generality, we may assume that $b \neq s^*, t^*$. Let $v$ be a vertex of $V(F) - X' - Y' - \{s^*, t^*\}$. Now consider the linkage problem

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{\{v, x\} | x \in X'\} \cup \{\{v, b\}, \{s^*, t^*\}\}.$$ 

The linkage problem $\mathcal{L}$ has at most $\left(\frac{k}{2}\right) + k(k-1) + k + 2 \leq 2k^2$ pairs, and so there exists a solution $R$ in $F$. Let $R \in \mathcal{R}$ be the path with ends $s^*$ and $t^*$. We now define $P^*$ to be the shortest induced subpath of $sPs's^*Rt'Pt$. We claim that $P^*$ is the desired path violating our choice of $P$. Let $S_i^* = S_i(P^*)$ for $i = 0, \ldots, k$.

To complete the proof, it now suffices to verify the following claim.

**Claim 1** $(S_k^*, \ldots, S_0^*)$ is lexicographically greater than $(S_k, \ldots, S_0)$

**Proof.** We begin with the observation that by construction and the choice of $s^*$ and $t^*$, there exists a subpath $\overline{R}$ of $R$ with ends $\overline{s}$ and $\overline{t}$ such that $P^* = sPs's^*Rt'Pt$. Furthermore, it follows that $E(P[A]) \supseteq E^*(P[A])$ and $E(P^*) - E(P) \subseteq E(F) \cup \{s^*, t^*\}$. It follows that $E(P^*) \cap E(S) = \emptyset$ since the edges $s^*\overline{s}$ and $t^*\overline{t}$ each have at least one endpoint in $F$ and $F$ and $S$ are disjoint.

For any vertex $u \in V(G)$ such that $u \in S_i$ for some $i > \text{min}$, it suffices now to show that $u$ has $i$ internally disjoint paths from $u$ to distinct vertices in $S$ to imply that $u \in S_j^*$ for some $j \geq i$. To see this, first observe that the vertex $u$ must be contained in $A$. Assume as a case that $u \in A - X$. In the graph $G - E(P)$, there exist $i$ internally disjoint paths $N_1, \ldots, N_i$ each with a distinct end in $S$ and the other endpoint equal to $u$. Then any path $N_i$ with at most one vertex in $X$ does not contain any edge of $(G - E(P))|B)$ and consequently does not use any edges of $P^*$. Any path $N_i$ that does use at least two vertices of $X$ has a first and last vertex in $X$. There exists a path from $X$ to $X'$ avoiding the edges of $P^*$, and consequently a path in $\mathcal{R}$ connecting the ends in $X'$ avoiding edges of $P^*$. It follows that $u \in S_j^*$ for some $j \geq i$.

We now assume $u \in X$. One path from $u$ to $S$ can be found as above by following the linkage from $X$ to $X'$ and using a path in the solution to the linkage problem $\mathcal{L}_1$. However, as many as $i$ of the paths ensuring that $u \in S_i$ may have used edges contained in $B - A$. Thus the solution to the linkage problem $\mathcal{L}_2$ will ensure that $u$ has $i$ internally disjoint paths to distinct vertices in $S$ in $G - E(P^*)$. Let $Q_1^u, \ldots, Q_i^u$ be the internally disjoint paths linking $u$ to distinct vertices of $S$ contained in $Q$. As in the previous paragraph, any path that uses at most one vertex of $V(F)$ will still exist in $G - E(P^*)$. If $Q_i^u$ uses at least two vertices of $V(F)$, then by the fact that $\mathcal{R}$ contains a solution to the linkage problem $\mathcal{L}_2$, there exists a path of $\mathcal{R}$ rerouting $Q_i^u$ to avoid any edge of $P^*$.

We now will see that the vertex $v \in V(F)$ lies in $S_j^*$ for some $j > \text{min}$. The vertex $v$ has $|X|$ internally disjoint paths in $F$ to $X'$ that avoid $E(P^*)$ and an additional path to the vertex $b$. Then $X'$ is linked to $X$ avoiding $E(P)$, and as a consequence, avoiding $E(P^*)$. Furthermore, by construction, the edge $bb'$ is not contained in $E(P^*)$. Finally, our choice of separation $(A, B)$ ensures that $X \cup \{b'\}$ sends $|X| + 1$ disjoint paths to $V(S)$ avoiding edges of $P^*$ to prove that $v \in S_j^*$ for some $j > \text{min}$. This completes the proof of the claim. \qed

This completes the proof of Theorem 1.3.
3 An Approach to Conjecture 1.1

We make the following conjecture:

**Conjecture 3.1** There exists a function $f = f(k)$ such that the following holds. Let $G$ be an $f(k)$-connected graph and let $s$, $t$ and $v$ be three distinct vertices of $G$. Then $G$ contains an $s - t$ path $P$ and a $k$-connected subgraph $H$ such that $v \in V(H)$ and furthermore, $H$ and $P$ are disjoint.

We will see that Lovász’ conjecture in fact follows from Conjecture 3.1

**Theorem 3.2** If Conjecture 3.1 is true, then Conjecture 1.1 is true.

**Proof.** Let $f(k)$ be a function satisfying Conjecture 3.1. We show the existence of a function $g(k)$ satisfying Conjecture 1.1, where $g(k)$ will be any function sufficiently large to make the necessary inequalities of the proof true.

Let $s$ and $t$ be two fixed vertices of a $g(k)$-connected graph $G$, and let $F$ be a maximal $k$-connected subgraph that does not separate $s$ and $t$. To see that such a subgraph $F$ must exist, consider a shortest path $P$ from $s$ to $t$. Every vertex not contained in $P$ can have at most three neighbors on $P$, and so the minimum degree of $G - V(P)$ must be strictly greater than $4k$. Theorem 1.5 implies that there exists a $k$-connected subgraph that does not separate $s$ and $t$.

A block is a maximal 2-connected subgraph. Every connected graph $G$ has a block decomposition $(T, B)$ where $T$ is a tree and $B = \{B_v | v \in V(T)\}$ is a collection of subsets of vertices of $G$ indexed by the vertices of $T$ such that the following hold:

i. for every $v \in V(T)$, $G[B_v]$ is either an edge or a block of $G$,

ii. for every edge $uv$ of $T$, $|B_v \cap B_u| = 1$, and

iii. every edge of $G$ is contained in $B_v$ for some $v \in V(T)$.

Observe that for any edge $uv \in E(T)$, the vertex in $B_u \cap B_v$ is a cut vertex of the graph. See [2] for more details.

Consider a block decomposition $(T, B)$ of the component of $G - F$ containing $s$ and $t$. Assume there exists a leaf $v$ of $T$ such that such that $B_v - u$ does not contain either $s$ or $t$ (where the vertex $u$ separates $B_v - \{u\}$ from the rest of $G - F$). Then deleting any vertex of $B_v - \{u\}$ does not separate $s$ and $t$. If any such vertex $x$ in $B_v - \{u\}$ had $k$ neighbors in $F$, then $F \cup x$ would be a $k$-connected graph that does not separate $s$ and $t$, contradicting our choice of $F$. It follows that $G[B_v - \{u\}]$ has minimum degree at least $g(k) - k$. We assume $g(k)$ satisfies the following inequality:

$$g(k) - k \geq 4k^2.$$ 

By Theorem 1.6, we conclude $G[B_v - u]$ has a $k$-connected subgraph $H$ whose boundary has at most $2k^2$ vertices. It follows that there exists a matching of size at least $k$ from $V(H) - \partial_{G[B_v]}(H)$ to $V(F)$ in $G$. This is a contradiction, since then $H \cup F$ is a larger $k$-connected subgraph that does not separate $s$ from $t$.

By the same argument as above, $G - F$ has exactly one component. It follows that the block decomposition $(T, B)$ of $G - F$ has $T$ equal to a path. Let the blocks of the decomposition be $B_0, \ldots, B_l$ with $B_i \cap B_{i+1} = v_i$. Then we may assume that $s \in B_0$ and $t \in B_l$. Moreover, for all $i = 0, \ldots, l - 1$, it follows that $v_i \neq v_{i+1}$, and $s \neq v_0$ and $t \neq v_{l-1}$.
Now assume there exists a block $B_i$ which is non-trivial, i.e., not a single edge. Let $s' = s$ if $i = 0$, and $s' = v_{i-1}$ otherwise. Similarly, let $t' = t$ if $i = l$ and $t' = v_i$ otherwise. Observe that any vertex $v$ of $B_i - \{s', t'\}$ does not separate $s'$ from $t'$, and so, as above, $v$ cannot have more than $k$ neighbors in $F$, lest we contradict our choice of $F$. It follows that $G[B_i - \{s', t'\}]$ has minimum degree at least $g(k) - k - 1$. We assume that
\[ g(k) - k - 1 > 4f(k+1)^2. \]

Then $G[B_i - \{s', t'\}]$ contains an $f(k+1)$-connected subgraph $F'$ with boundary at most $2f(k+1)^2$. Moreover, by the connectivity of $G$, there exist $f(k+1)$ vertices $u_1, \ldots, u_{f(k+1)} \in V(F') - \partial G[B_i - \{s', t'\}](F')$ such that each has a distinct neighbor in $F$ (in the graph $G$).

Attempt to find a path from $s'$ to $t'$ in $G[B_i - V(F')]$. If such a path exists, then $F'$ does not separate $s'$ from $t'$ in $G[B_i]$, and the subgraph induced by $V(F \cup F')$ contradicts our choice of $F$ to be as large as possible. It follows that $F'$ does separate $s$ from $t$ in $G - F$. Let $\overline{P}$ be a path in $G[B_i]$ with ends $s'$ and $t'$. Let $\overline{v}$ be the vertex of $V(\overline{P}) \cap V(F')$ closest to $s'$ on $\overline{P}$. Similarly, let $\overline{t}$ be the vertex of $V(\overline{P}) \cap V(F')$ closest to $t'$ on $\overline{P}$. We define a new graph $\overline{F}$ with vertex set $V(\overline{F})$ equal to $V(F') \cup \overline{v}$ where $\overline{v}$ is a new vertex representing the subgraph $F$. The edge set of $\overline{F}$ is given by $E(\overline{F}) = E(F') \cup \{\overline{v}u_i|i = 1, \ldots, f(k+1)\}$. Then $\overline{F}$ is an $(f(k+1))$-connected graph, so by our assumption that $f$ is a function satisfying Conjecture 3.1, there exists a $(k+1)$-connected subgraph $H$ of $\overline{F}$ containing the vertex $\overline{v}$, and moreover, $F' - H$ contains a path from $\overline{s}$ to $\overline{t}$. By construction, $H - \overline{v}$ is a $k$-connected subgraph of $G[B_i]$ that does not separate $s$ from $t$, and moreover, there exists a matching of size $k$ from $H - \overline{v}$ into the vertices of $F$. It follows that $G[V(F) \cup V(H) - \{\overline{v}\}]$ is a subgraph violating our choice of $F$ to be a maximum $k$-connected subgraph not separating $s$ from $t$. This contradicts our assumption that the block decomposition of $G - F$ contained a non-trivial block. It follows that $G - F$ is an induced $s - t$ path, completing the proof.

Conjecture 3.1 is closely related to the following strengthening of Conjecture 1.1 due to Thomassen.

**Conjecture 3.3 (Thomassen, [15])** For every $l, t \in \mathbb{N}$ there exists $k = k(l, t) \in \mathbb{N}$ such that for all $k$-connected graphs $G$ and $X \subseteq V(G)$ with $|X| \leq t$, the vertex set of $G$ can be partitioned into non-empty sets $S$ and $T$ such that $X \subseteq S$, each vertex in $S$ has at least $l$ neighbors in $T$ and both $G[S]$ and $G[T]$ are $l$-connected subgraphs.

As the conjecture originally appeared, $t$ was assumed to be equal to $l$. We have introduced the additional parameter to discuss partial progress on the conjecture.

**Observation 3.4** If $\forall l \geq 0, 0 \leq t \leq 2$ there exists a positive integer $k = k(l, t)$ satisfying Conjecture 3.3, then Conjecture 1.1 is true.

**Proof.** Let $l$ be any positive integer, $k = k(l, 2)$ be as in Conjecture 3.3, and let $G$ be a $k$-connected graph. Then there exists a partition $(A, B)$ of the vertices of $G$ such that $s, t \in A$, $G[A]$ and $G[B]$ are $l$-connected graphs, and, furthermore, every vertex of $A$ has at least $l$ neighbors in $B$. Then if $P$ is a path in $G[A]$ connecting $s$ and $t$, $G - V(P)$ is an $l$-connected graph. Thus $f(l) = k(l, 2)$ is a function satisfying Conjecture 1.1.

Kühn and Osthus [3] have proven Conjecture 3.3 is true when the integer $t$ is restricted to 0. A consequence of Theorem 3.2 is the following corollary.
Corollary 3.5 If $\forall l \geq 0, 0 \leq t \leq 1$ there exists a positive integer $k = k(l, t)$ satisfying Conjecture 3.3, then Conjecture 1.1 is true.

Proof. Let $l$ be a positive integer and let $k = k(l + 2, 1)$ be the value given by Conjecture 3.3. Then let $G$ be a $k$-connected graph, and let $v, s,$ and $t$ be given as in Conjecture 3.1. Let $(A, B)$ be a partition of $V(G)$ such that $G[A]$ and $G[B]$ are $(l+2)$-connected, and furthermore, that $v \in A$. Then $G[A - \{s, t\}]$ is an $l$-connected subgraph containing $v$ that does not separate $s$ and $t$, as desired. □

References