Extremal functions for rooted minors

Paul Wollan *†

Abstract

The graph G contains a graph H as a *minor* if there exist pair-wise disjoint sets $\{S_i \subseteq V(G) | i = 1, \ldots, |V(H)|\}$ such that for every i, $G[S_i]$ is a connected subgraph and for every edge uv in H, there exists an edge of G with one end in S_u and the other end in S_v. A *rooted* H minor in G is a minor where each S_i of minor contains a predetermined $x_i \in V(G)$. We prove that if the constant c is such that every graph on n vertices with cn edges contains a H minor, then every $|V(H)|$-connected graph G with $(18c + 1236|V(H)|)|V(G)|$ edges contains a rooted H minor for every choice of vertices $\{x_1, \ldots, x_{|V(H)|}\} \subseteq V(G)$. The proof methodology is sufficiently robust to find the exact extremal function for an infinite family of rooted bipartite minors previously studied by Jorgensen, Kawarabayashi, and Böhme and Mohar.

1 Introduction

The graph G contains a graph H as a *minor* if there exist pair-wise disjoint sets $\{S_i \subseteq V(G) | i = 1, \ldots, |V(H)|\}$ such that for every i, $G[S_i]$ is a connected subgraph and for every edge uv in H, there exists an edge of G with one end in S_u and the other end in S_v. This definition differs slightly from the more traditional definition based on edge contractions and deletions, but is more suited for our purposes. The S_i will be referred to as the *branch sets* of the H minor in G. A classical question of graph theory asks how many edges must a graph G have, as a function of the number of vertices $|V(G)|$, in order to ensure that G contains some fixed graph H as a minor. Thomason [13] and Kostochka [8] independently proved that there exists a constant c such that every graph on n vertices with $ct\sqrt{\log t}n$ edges contains K_t as a minor.

We will focus on a variant of this question where we fix some set X of the vertices of G, and then ask how many edges must G have in order to have a H minor where each branch set of the minor contains a pre-specified vertex $x \in X$. Rigorously, we define a π-rooted H minor as follows.

Definition Let G and H be graphs and $X \subseteq V(G)$ with $|X| = |V(H)|$. Let $\pi : X \to V(H)$ be a bijection. Then the pair (G, X) contains a *π-rooted H minor* if there exist $\{S_i | i = 1, \ldots, |V(H)|\}$ forming the branch sets of an H minor such that for every $x \in X$, $x \in S_{\pi(x)}$.

In order to avoid trivial counterexamples, any good edge bound to ensure that a pair (G, X) contains a specified rooted H minor will require that the graph G be at least $|H|$ connected. Otherwise, G could be almost complete but if the set X is separated from all of G by a small cut set, (G, X) would not contain the rooted H minor. The following theorem shows that if we do assume a minimal amount of connectivity, then the number of edges required to ensure that (G, X) contains a π-rooted H minor for every map π is only a constant factor more than the number of edges necessary to ensure that G contains a H minor in the first place.

Theorem 1.1 *Let H be a fixed graph and $c \in \mathbb{R}$, $c \geq 1$ be a constant such that every graph on n vertices with at least cn edges contains H as a minor. If G is any graph such that G is $|V(H)|$ connected and has at least $(18c + 1236|V(H)|)|V(G)|$ edges, then for all sets $X \subseteq V(G)$ with $|X| = |V(H)|$ and for all bijective maps $\pi : X \to V(H)$, (G, X) contains a π-rooted H minor.*

*School of Mathematics Georgia Institute of Technology Atlanta, Georgia 30332 USA

†Email address: wollan@math.gatech.edu

Notice that the results of Thomason and Kostochka cited above immediately imply the existence of such a constant c.

Several specific families of rooted minors have been studied previously. A graph is *k-linked* if for any $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ there exist disjoint paths $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i for every i. If we let H be the graph consisting of $2k$ vertices and k disjoint edges, then the property of being k-linked is equivalent to (G, X) containing a π-rooted H minor for every set X of $2k$ vertices and every bijective map $\pi : X \to V(H)$. For a graph G to contain an H minor, we only need sufficiently many edges to guarantee that the G contains k distinct edges that pairwise do not share an endpoint. It is easy to see that $2k|V(G)|$ edges suffice to ensure this property. Up to the multiplicative constant, this is best possible. Consider a graph with n vertices and $k-1$ vertices adjacent to every other vertex. Such a graph would have $(k-1)(n-1) - \binom{k-1}{2}$ edges and does not contain k disjoint edges. Bollobás and Thomason proved in [2] that every $2k$-connected graph with $11k|V(G)|$ edges is k-linked. Theorem 1.1 similarly proves that when we allow H to be an arbitrary graph, again, we only need a constant factor more edges to ensure that we can find the H minor with every branch set containing a pre-specified vertex of G. An improvement to the result of Bollobás and Thomason that we will use in later sections is the following:

Theorem 1.2 *([12]) Every $2k$-connected graph G with at least $2k$ vertices and $5k|V(G)|$ edges is k-linked.*

The study of graph linkages has recently been generalized to examine H-linkages. Given a fixed graph H, a graph G is *H-linked* if for any specified $v_1, \ldots, v_{|V(H)|}$ vertices of G, there exists an H subdivision in G with the specified v_i's as the branch vertices. Kostochka and Yu [9, 10] and Ferrara, Gould, Tansey, Whalen [4] and Gould and Whalen [5, 14] have quantified exact minimal degree conditions that force a graph to be H-linked. Asking a graph to be H-linked requires the existence of a topological H minor sitting on a predetermined set of vertices. Rooted H minors generalize this further by asking for a minor instead of a topological minor to sit on the specified vertices.

Section 2 discusses a general approach to finding an extremal function for a particular rooted minor. Theorem 1.1 follows this approach to get a bound for containing an arbitrary H as a rooted minor. The method is sufficiently robust, however, that when we restrict to a single family of rooted structures, we can get the exact extremal function.

Definition Let G be a graph and $X \subseteq V(G)$ with $|X| = t$. Then the pair (G, X) contains a $K_{2,t}(X)$ if there exist pair wise disjoint sets $A_1, \ldots, A_t, B_1, B_2 \subseteq V(G)$ such that every A_i and B_i induces a connected subgraph of G and for every $i = 1, \ldots, t$ and $j = 1, 2$, there exists an edge of G with one end in A_i and the other end in B_j. Moreover, each A_i contains exactly one vertex of X.

The structure $K_{2,t}(X)$ was first examined by Jorgensen in [11] in the case $t = 4$. Jorgensen found the the exact extremal function as a lemma while calculating the extremal function for $K_{4,4}$ minors. This work was later extended to $K_{3,4}(X)$ in a paper by Kawarabayashi and Jorgensen in [7]. Kawarabayashi in [6] examined connectivity constraints for forcing arbitrary rooted $K_{a,k}(X)$ minors, however the results do not give optimal bounds. The $K_{2,t}(X)$ minors were independently studied by Böhme and Mohar in [1], where the authors referred to them as *labeled* $K_{2,t}$ minors.

We prove the following optimal edge bound for $K_{2,t}(X)$ minors.

Theorem 1.3 *Let G be a t-connected graph on n vertices and let $X \subseteq V(G)$ with $|X| = t$. If $|E(G)| \geq tn - \binom{t+1}{2} + 1$, then G contains a $K_{2,t}(X)$.*

The edge bound in Theorem 1.3 is the best possible. Consider a graph G with vertex set $\{x_1, \ldots, x_t, v_1, v_2, \ldots, v_n\}$ and the following edges: x_i is adjacent x_j for all i and j. The vertex v_1 is adjacent x_i for $i = 1, \ldots, t$. For vertices v_i with $i \geq 2$, v_i is adjacent x_j for every $j \geq 2$ and v_i is adjacent v_{i-1}. Then if $X := \cup_{i=1}^{t} x_i$, G does not contain a $K_{2,t}(X)$. However, G is t-connected and has $t|V(G)| - \binom{t+1}{2}$ edges.

For graph-theoretic terminology not explained in this paper, we refer the reader to [3]. Given a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. For a subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$ and $N_G(S) = \bigcup_{s \in S} N_G(s) - S$. For a subgraph H of G, $G - H = G[V(G) - V(H)]$, and for a vertex x of $V(G)$ and for an edge e of E(G), $G - x = G[V(G) - \{x\}]$ and $G - e$ is the graph obtained from G by deleting e.

2 A General Technique

A difficulty when looking for a sufficient edge bound to force the kinds of rooted minor structures above is that any reasonable edge bound will require some basic amount of connectivity. In order to apply induction, we would like to be able to contract and remove edges; but this can cause problems because in general connectivity is not maintained under edge contraction or deletion. Towards this end, we define a suitable relaxation of connectivity that is conducive to inductive proofs.

For notation, in a graph G and a set $X \subset V(G)$, let $\rho(X)$ be the number of edges with at least one end in X. A *separation* of a graph G is a pair of subsets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$ and no edge $uv \in E(G)$ has $u \in A - B$ and $v \in B - A$. A separation (A, B) of a graph G where $A \subseteq B$ is *trivial*. Unless otherwise stated, all separations in this article are assumed to be non-trivial.

Definition Let G be a graph, let $X \subseteq V(G)$, and let $\lambda > 0$ be a real number. We say that the pair (G, X) is λ-*massed* if
(M1) $\rho(V(G) - X) > \lambda|V(G) - X|$, and
(M2) every separation (A, B) of (G, X) of order at most $|X| - 1$ satisfies $\rho(B - A) \leq \lambda|B - A|$.

We fix a set X as a reference; in the following theorems, X will be the set onto which we are trying to root our minor. Now we only care about small separations that split pieces of the graph away from the set X. Then condition (M2) allows such small separations as long as they do not separate off too many edges. Condition (M1) then ensures that globally, the graph has many edges. Since the majority of the edges are not separated from X by small cut-sets, the graph is in some sense *massed* around the set X.

Assume now that we have a fixed graph H, and we are trying to show that a constant α suffices so that an α-massed pair (G, X) has a π-rooted H minor for an arbitrary injective functions π from X to $V(H)$. We now rigorously define what we mean by a minimal pair (G, X) not containing such a rooted H-minor.

Definition Let H be some fixed graph and α a positive real number. Let G be a graph and X a subset of $V(G)$ such that

1. $|X| \leq |V(H)|$,
2. (G, X) is α-massed,
3. and there exists an injection $\pi : X \to V(H)$ such that if H' is the image of π, then (G, X) does not contain a π rooted H' minor.
4. Subject to (1.), (2.), (3.), $|V(G)|$ is minimal.
5. Subject to (1.), (2.), (3.), and (4.), $\rho(G - X)$ is minimal.

Then we say that (G, X) is (H, α)-*minimal*.

Given an (H, α) minimal pair (G, X), we see that when we attempt to contract an edge of the graph G not connecting two vertices of X, we violate condition (M1) or (M2) to prevent $(G/e, X)$ from being α-massed. This naturally leads to separations (A, B) in G where $X \subseteq A$ and for every injection $\overline{\pi} : A \cap B \to V(H)$, $(G[B], A \cap B)$ contains a $\overline{\pi}$-rooted $\overline{H}$ minor where $\overline{H}$ is the subgraph induced by the image of $\overline{\pi}$. We will call such separations *rigid*. Rigorously, we define the following.

Definition Let G and H be graphs and $X \subseteq V(G)$. Then a separation (A, B) is a H-*rigid* separation of the pair (G, X) if

1. $X \subseteq A$,

2. the order of (A, B) is at most $|V(H)|$,

3. and the pair $(G[B], A\cap B)$ contains a π-rooted H' minor for all subgraphs H' of H with $|V(H')| = |A\cap B|$ and for all injections $\pi : A \cap B \to V(H')$.

An advantage of rigid separations is they allow us to reduce finding a π-rooted H' minor in (G, X) to finding a π-rooted H' minor in a smaller graph. We prove:

Theorem 2.1 *Let H and G be graphs and $X \subset V(G)$. If (G, X) is (H, α)-minimal, then (G, X) does not contain an H-rigid separation.*

Upon eliminating rigid separations from our pair (G, X), we are now in a position to apply the more traditional tricks for proving extremal functions for graph minors. Specifically, we reduce the problem to examining a small dense neighborhood of the graph G.

Theorem 2.2 *Let G and H be graphs and α positive real number with $\alpha \geq |V(H)|$. If (G, X) is (H, α)-minimal for a set of vertices $X \subset V(G)$, then the following hold:*

1. *Let uv be an edge of G not contained in X. Then if neither u nor v is in X, u and v have at least $\lfloor \alpha \rfloor$ common neighbors. If one, say u is an element of X, then if v has exactly t neighbors in X other than u, u and v have at least $\lfloor \alpha \rfloor - t$ common neighbors.*

2. *There exists a vertex in $V(G) - X$ of degree at most 2α.*

One strategy to utilize this small dense neighborhood is to find a subgraph D in the neighborhood such that for any set X' of D with $|X'| = |V(H)|$ such that (D, X') contains a π-rooted H minor for every choice of injective map $\pi : X' \to V(H)$. Then Theorem 2.3 gives a final contradiction to the fact that we chose (G, X) to be an (H, α) minimal pair.

Definition Given graphs G and H with $|V(G)| \geq |V(H)|$, G is *H-universal* if for all subsets $X \subset V(G)$ with $|X| = |V(H)|$ and for all injective maps $\pi : X \to V(H)$, (G, X) contains a π-rooted H minor.

Theorem 2.3 *Given graphs G and H and a positive real number α, if (G, X) is (H, α)-minimal for some set of vertices $X \subset V(G)$, then G does not contain an H universal subgraph.*

3 Proof of Theorem 2.1

Before proving Theorem 2.1, we first prove a lemma about matchings in bipartite graphs with specific properties.

Definition Let G be a bipartite graph with bipartition (X, Y). A *neighborhood cover matching* is a pair $(\mathcal{M}, \kappa)$ where $\mathcal{M}$ is a matching in G and κ is a 2 coloring of the edges of $\mathcal{M}$ such that for every vertex $x \in X$ one of the following conditions holds:

1. either for all $y \in N(x)$ there exists a z with $yz \in \mathcal{M}$ and $\kappa(yz) = 1$, or

2. there exists a y in Y such that xy is in $\mathcal{M}$ and $\kappa(xy) = 2$.

If G has no edges, we allow the empty matching to be a neighborhood cover matching.

Lemma 3.1 *There exists a neighborhood cover matching for any bipartite graph $G = (X, Y)$.*

Proof. The proof is by induction on $|X|$. Clearly, if $|X| = 1$, then an arbitrary edge incident the vertex of X with that edge colored 2 forms a neighborhood cover matching.

If G contains a matching $\mathcal{M}$ covering X, then again, that $(\mathcal{M}, \kappa)$ where $\kappa(e) = 2$ for every edge of $\mathcal{M}$ forms a neighborhood covering matching. If G does not contain such a matching, then there exists a set $B \subseteq X$ violating the condition for Hall's Theorem. Assume B is such a set of minimal size. Then there exists a matching covering $N(B)$ in $G[B \cup N(B)]$, call it $\mathcal{M}_1$. If such a matching did not exist, then there would exist a set $J \subseteq N(B)$ violating Hall's condition in $G[B \cup N(B)]$. But then $B - N(J)$ would be a smaller set than B violating Hall's condition in G, contrary to our choice of B.

Now consider the subgraph G' induced by $(X - B, Y - N(B))$. By induction there exists a neighborhood cover matching $(\mathcal{M}_2, \kappa)$ (possibly the empty matching). Then we define the coloring κ' on $\mathcal{M}_1 \cup \mathcal{M}_2$ to create a neighborhood cover matching in G.

$$\kappa'(e) = \begin{cases} 1 & \text{if } e \in \mathcal{M}_1 \\ \kappa(e) & \text{if } e \in \mathcal{M}_2 \end{cases}$$

Let x be a vertex in $X - B$. Then if there is no edge e in $\mathcal{M}_\in$ incident x with $\kappa'(e) = 2$, then every neighbor of x in $Y - N(B)$ is incident an edge e of $\mathcal{M}_2$ with $\kappa'(e) = 1$. Moreover, since every vertex in $N(B)$ is incident an edge e of $\mathcal{M}_1$ with $\kappa'(e) = 1$, we see that every neighbor of x is incident a matching edge of color 1. Every vertex $x \in B$ has $N(x) \subseteq N(B)$, implying that every neighbor of x is incident an edge e with $\kappa'(e) = 1$. Thus every vertex in X satisfies the conditions of the definition, proving that G contains a neighborhood covering matching. □

Given a graph G and a separation (A, B) of G, we say the *separation truncation* of G is the graph G' equal to $G[A]$ with added edges between every pair of non-adjacent vertices of $A \cap B$. We now prove a lemma stating that the existence of rooted minors is preserved under taking separation truncations when the separation is rigid.

Lemma 3.2 *Let G and H be graphs and let $X \subseteq V(G)$ such that $|X| = |V(H)|$. Assume that (A, B) is an H rigid separation. Then for all injections $\pi : X \to V(H)$, (G, X) contains a π-rooted H minor if and only if (G', X) contains a π-rooted H minor where G' is the separation truncation of G with respect to (A, B).*

Proof. Let G, H, X and (A, B) be given as in the statement of the Lemma. Fix our map $\pi : X \to V(H)$, and let G' be the separation truncation of G with respect to (A, B).

First, we see that if (G, X) has a π rooted H minor, then (G', X) does as well. Let $\{S_1, \dots, S_k\}$ be the branch sets of a π-rooted H minor. Let $S_i' = S_i \cap A$. Clearly, the S_i' induce connected subgraphs because $G'[A \cap B]$ is complete. Moreover, let uv be an edge connecting S_i and S_j in G. If $uv \subseteq A$, then the edge is present between S_i' and S_j' in G'. Otherwise, both S_i and S_j intersect $A \cap B$, and so because $A \cap B$ induces a complete subgraph of G', there is an edge connecting S_i' and S_j', as desired. Thus the $\{S_1', \dots, S_k'\}$ do in fact form the branch sets of a π-rooted H minor in (G', X)

Now to prove the other direction, assume that (G', X) contains a π-rooted H minor, and further that we pick such a rooted minor to minimize the number of vertices in the bags. Let $\{S_1, \dots, S_k\}$ be the branch sets of the minor, so that x_i is a member of $S_{\pi(i)}$. At the expense of slightly confusing notation, for every S_i that intersects $A \cap B$ in at least two vertices, let P_i be a path from $x_{\pi^{-1}(i)}$ to $A \cap B$ in $G'[S_i]$. Let v_i be the end of P_i in $A \cap B$, and let $u_1^i, \dots, u_{t(i)}^i$ be the other vertices of S_i other than v_i in $A \cap B$. For every S_i with $|S_i \cap (A \cap B)| \geq 2$, let T^i be a spanning tree of $G'[S_i]$ with the following properties:

1. P_i is a subgraph of T^i, and
2. $T^i[A \cap B]$ is a star with root v_i and $u_1^i, \dots, u_{t(i)}^i$ forming the leaves of the star.

For every S_i that intersects $A \cap B$ in exactly one vertex, let v_i be $S_i \cap (A \cap B)$.

To prove that (G, X) contains a π-rooted H minor, it would now suffice to prove that edges of the form $v_i u_j^i$ and $v_i v_k$ for all appropriate i, j, and k can be reconstructed in G by choosing the right rooted minor H' on $(G[B], A \cap B)$ for some subgraph H' of H. Unfortunately, we must proceed more cautiously.

If we remove the edges $v_i u_1^i, \dots, v_i u_{t(i)}^i$ from T^i, the induced components of T^i partition the vertices of T^i into $t(i)+1$ subtrees. Let $T(v_i)$ be the subtree containing v_i and $T(u_j^i)$ be the subtree containing u_j^i. It is possible that $T(u_j^i)$ will simply be a trivial tree consisting of one vertex.

By the minimality of the number of vertices in branch sets, we know that for every defined $T(u_j^i)$, there is an edge going to some other T^l with l adjacent to i in H and $T^l \cap (A \cap B) = \varnothing$. There may in fact be several other branch sets of the minor to which T^i connects through $T(u_j^i)$. We define $\mathcal{N}(T(u_j^i))$ to be the set of all such indices l, or

$$\mathcal{N}(T(u_j^i)) = \left\{ l \in V(H) : \begin{array}{l} i \text{ is adjacent to } l \text{ in } H, \\ \text{there is an edge from } T(u_j^i) \text{ to } T^l, \\ \text{and } T^l \cap A \cap B = \varnothing \end{array} \right\}$$

Note that for any index $l \in \mathcal{N}(T(u_j^i))$, there exists a path from u_j^i to T^l using only vertices of $T(u_j^i)$ and one endpoint in T^l. For notation, denote such a path $P(u_j^i \to l)$.

Now consider the bipartite graph on the vertex set $W = \{u_j^i | i \in V(H), 1 \le j \le t(i)\} \cup Y = \{l | l \in V(H)\}$ where (W, Y) is the bipartition of the graph. The edges of the bipartite graph are given by $u_j^i l$ for all $l \in \mathcal{N}(T(u_j^i))$. Then we know from Lemma 3.1 that there exists a neighborhood cover matching from W to Y. For notation, we represent the matching as an injective function $\lambda : W \to V(H) \times \{1, 2\}$. Let λ_1 be the value λ takes on $V(H)$, and let λ_2 be the value λ takes on $\{1, 2\}$.

We are now ready to start constructing the branch sets of our π-rooted H minor in (G, X). We first pick an appropriate rooted minor of $(G[B], A \cap B)$. To pick our injective function $\phi : A \cap B \to V(H)$, we define ϕ as follows for every $x \in A \cap B \cap \{S_i : i = 1, \dots, |V(H)|\}$:

$$\phi(x) = \begin{cases} i & \text{if } x = v_i \text{ for some } i \\ \lambda_1(u_j^i) & \text{if } x = u_i^j \text{ for some } i \text{ and } j \end{cases}$$

To see that ϕ is an injection, assume that we have x and y such that $\phi(x) = \phi(y)$. Since we know that λ_1 is an injection by definition, and since $v_m \neq v_n$ for $n \neq m$, we may assume that $x = v_l$ for some l, and $y = u_j^i$ for some i and j. But this implies that $v_m \in \mathcal{N}(T(u_j^i))$, contrary to the definition.

Let H' be the subgraph of H induced on $Im(\phi)$. By definition of a rigid separation, we know there exists a ϕ-rooted H' minor in $(G[B], A \cap B)$. Let $\{U_i | i \in V(H')\}$ be the branch sets of the rooted minor where $x \in A \cap B$ is an element of $U_{\phi(x)}$. There is a slight abuse of notation here in that rooted minors of (G, X) are only defined for injections from X, where as ϕ is not be defined for the vertices of $A \cap B$ not in any S_i. However, in such a case, ϕ could be arbitrarily defined for the remaining vertices of $A \cap B$. We will only need the branch sets of the H' minor rooted on the original domain of ϕ.

We now define the branch sets $\overline{S}$ forming a π-rooted H minor in (G, X). For i with $|S_i \cap (A \cap B)| \ge 2$, let

$$\overline{S_i} = V\left(T(v_i)\right) \cup U_{\phi(v_i)} \bigcup_{\{j | \lambda_2(u_j^i) = 1\}} U_{\phi(u_j^i)} \cup V\left(T(u_j^i)\right).$$

When S_i intersects $A \cap B$ in exactly one vertex, let

$$\overline{S_i} = S_i \cup U_{\phi(v_i)}$$

Among the l such that S_l does not intersect $A \cap B$, there are two separate cases: when some u_j^i is mapped to l by λ, or not. For l such that there exists a $u_j^i \in A \cap B$ with $\lambda(u_j^i) = (l, 2)$

$$\overline{S_l} = S_l \cup P(u_j^i \to l) \cup U_{\phi(u_j^i)}.$$

Observe that by the fact that λ is a matching, there is at most one such u_j^i. Otherwise,

$$\overline{S_i} = S_i.$$

For any $x \in X$, if $|S_{\pi(x)} \cap (A \cap B)| \geq 2$, $T(v_{\pi(x)}) \subseteq \overline{S}_{\pi(x)}$. Otherwise, $S_i \subseteq \overline{S_i}$. Clearly, then, $x \in \overline{S}_{\pi(x)}$. In order to check that the $\overline{S_i}$'s form the branch sets of a π rooted minor in (G, X), we simply need to verify they induce pair-wise disjoint connected subgraphs and that for any edge xy in H, there is an edge between $\overline{S_x}$ and $\overline{S_y}$.

By construction and the fact that ϕ is an injection, we see that the $\overline{S_i}$'s are pair-wise disjoint. Now we confirm that the $\overline{S_i}$ induce connected subgraphs. Observe that for every i, j, and k, the sets $T(v_i)$, $T(u_j^i)$, $U_{\phi(x)}$, and $P(u_j^i \to l)$ induce connected subgraphs of G. We conclude that if S_l intersects $A \cap B$ in at most 1 vertex, then the sets comprising $\overline{S_l}$ intersect so that their union again induces a connected subgraph.

Instead, assume $|S_l \cap (A \cap B)| \geq 2$. For every U_x with $U_x \subset \overline{S_l}$ and $x \neq v_l$, we know $U_x = U_{\lambda_1(u_j^l)}$, for some value of j, implying that $T(u_j^l) \cup U_x$ induce a connected subgraph. Moreover, $\phi(u_j^l) = \lambda_1(u_j^l) \in \mathcal{N}(u_j^l)$, implying that there is an edge between U_x and U_l. Every $U_x \cup T(u_j^l)$ induces a connected subgraph and is attached to U_l. U_l contains v_l, connecting it to $T(v_l)$. Thus $\overline{S_l}$ in fact induces a connected subgraph.

Now we prove that every edge of H is present between the appropriate branch sets of our prospective rooted minor. If the edge xy of H is such that S_x and S_y both intersect $A \cap B$ in at most one vertex, then $\overline{S_x} \supseteq S_x$ and $\overline{S_y} \supseteq S_y$. The only possible way that the edge xy could not be present in G is if both S_x and S_y intersect $A \cap B$ and the only edge in the separation truncation between S_x and S_y is the edge in $A \cap B$. However, in this case, $\overline{S_x} \supseteq U_x$ and $\overline{S_y} \supseteq U_y$, and so there is an edge between the two sets of vertices.

We now show that for any i with S_i such that $|S_i \cap (A \cap B)| \geq 2$ there exists an edge between $\overline{S_i}$ and every $\overline{S_l}$ with i adjacent l in H. Given such an $\overline{S_i}$, first we assume S_l intersects $A \cap B$ in at least one vertex. Then $\overline{S_l} \supseteq U_l$. Since $\overline{S_i} \supseteq U_i$ as well, then there is an edge between $\overline{S_l}$ and $\overline{S_i}$

Assume now that $S_l \cap (A \cap B) = \varnothing$. If the edge in the separation truncation between S_i and S_l is in fact an edge between $T(v_i)$ and S_l, then the edge is an edge of G and given that $\overline{S_i} \supseteq T(v_i)$, we know there is an edge between $\overline{S_i}$ and $\overline{S_l}$. And in fact, if the edge between S_l and S_i is an edge between S_l and some $T(u_j^i) \subseteq \overline{S_i}$, then there is an edge between $\overline{S_i}$ and $\overline{S_l}$. Thus we may assume that there is an edge between S_l and some $T(u_j^i) \nsubseteq \overline{S_i}$. Then $\lambda_2(u_j^i) = 2$ or u_j^i is not matched to any vertex of H in the neighborhood cover matching. However, now for every index in $\mathcal{N}(T(u_j^i))$, there is some other vertex of $A \cap B$ matched to it with an edge colored color 2 by the definition of a neighborhood cover matching. Specifically, there exists some $u_{j'}^{i'}$ with $\lambda(u_{j'}^{i'}) = (l, 2)$. Consequently, $\overline{S_l} \supseteq U_{\phi(u_{j'}^{i'})} = U_l$ and there exists an edge between $\overline{S_i}$ and $\overline{S_l}$, as desired.

This completes the proof that the set $\{\overline{S_i} | i \in V(H)\}$ form the branch sets of a π-rooted H minor of (G, X).

$\square$

Proof. (Theorem 2.1) Let (G, X), π, H, and H' be given as in the statement of Theorem 2.1 and in the definition of (H, α)-minimal. Assume, to reach a contradiction, that (G, X) has a rigid separation (A, B). Pick the separation (A, B) over all such rigid separations to minimize $|A|$. For notation, let $X = \{x_1, \dots, x_k\}$. We now proceed with several intermediate claims:

Claim 3.3 $|A \cap B| < |X|$

Proof. Assume otherwise. First consider when there exist k disjoint paths $P_1, \dots, P_k$, each with one end in X and the other in $A \cap B$. Without loss of generality, let the ends of P_i be x_i and a_i where $a_i \in A \cap B$. Then by the definition of rigid separation, $(G[B], A \cap B)$ contains a π-rooted H' minor with branch sets $\{S_1, \dots, S_k\}$ with $a_i \in S_{\pi(i)}$. Then (G, X) contains a π-rooted H' minor with branch sets $\{S'_1, \dots, S'_k\}$ where $S'_{\pi(i)} = S_{\pi(i)} \cup P_i$, a contradiction.

Thus no such disjoint paths exist, implying that $G[A]$ contains a separation (A', B') of order strictly less than k, with $X \subseteq A'$ and $A \cap B \subseteq B'$. But if we pick such a separation (A', B') of minimal order, then by the same argument as in the previous paragraph, $(A', B' \cup B)$ is a rigid separation. Moreover, $|A'| < |A|$, contrary to our assumptions.

$\square$

Let (G', X) be the separation truncation of (G, X) with respect to the separation (A, B).

Claim 3.4 *(G', X) is α-massed.*

Proof. By Claim 3.3 and condition $(M2)$ applied to (G, X), we see that (G', X) must satisfy condition $(M1)$. So assume that (G', X) contains a separation (A', B') violating condition $(M2)$, and assume we pick (A', B') to minimize $|B'|$. Then $(G'[B'], A' \cap B')$ is α-massed, and so by the fact that (G, X) is (H, α)-minimal, we know that in fact $(G'[B'], A' \cap B')$ contains a $\overline{\pi}$-rooted $\overline{H}$ minor for any subgraph $\overline{H}$ of size $|A' \cap B'|$ and any injective map $\overline{\pi}$ from $A' \cap B'$ to $V(\overline{H})$.

The subgraph of G' induced by $A \cap B$ is complete, and so it must be a subset of A' or B'. If $A \cap B \subseteq A'$, then $(A' \cup B, B')$ is a separation in G violating condition $(M1)$.

Thus we know $A \cap B \subseteq B'$. Then (B', B) is a rigid separation of $(G[B \cup B'], A' \cap B')$, and the separation truncation of $G[B' \cup B]$ with respect to the separation (B', B) is simply $G'[B']$. By (H, α) minimality, $(G'[B'], A' \cap B')$ contains a $\overline{\pi}$ rooted $\overline{H}$ minor. Lemma 3.2 then implies that $(G[B \cup B'], A' \cap B')$ contains a $\overline{\pi}$ rooted $\overline{H}$ minor. But this was for an arbitrary subgraph $\overline{H}$ of H, and an arbitrary map $\overline{\pi}$. Thus $(A', B' \cup B)$ is a rigid separation, contrary to our choice of (A, B) to minimize $|A|$.

□

Now by the definition of (H, α) minimality, (G', X) contains a π rooted H' minor. Lemma 3.2 implies that (G, X) contains a π-rooted H' minor, contrary to the fact that (G, X) is (H, α) minimal.

This completes the proof of Theorem 2.1. □

4 Proof of Theorems 2.2 and 2.3

Proof. (Theorem 2.2)

Let G, X, π, H, and H' be given as in the statement of the theorem and the definition of (H, α) minimality.

Claim 4.1 *Let uv be an edge of G not contained in X. Then if neither u nor v is in X, u and v have at least $\lfloor \alpha \rfloor$ common neighbors. If one, say u is an element of X, then if v has exactly t neighbors in X other than u, u and v have at least $\lfloor \alpha \rfloor - t$ common neighbors.*

Proof. Let uv be an edge not contained in X. Then if $(G/uv, X)$ has a π-rooted H' minor, (G, X) would as well. By minimality, we may assume, then, that $(G/uv, X)$ is not α-massed. Let (A, B) be a separation of $(G/uv, X)$ violating (M2), and assume that (A, B) is chosen to minimize $|B|$ from all such separations. Then $(G/uv[B], A \cap B)$ is α-massed. By the (H, α)-minimality of (G, X), the separation (A, B) of $(G/uv, X)$ is rigid. The separation (A, B) induces a separation (A^*, B^*) of (G, X). Let v_e be the vertex of G/uv corresponding to the contracted edge uv. If $v_e \in A$, let $A^* = A - v_e \cup \{u, v\}$, and $A^* = A$ otherwise. Similarly, define B^*. Notice that $\rho(B^* - A^*) \geq \rho(B - A)$. Because (G, X) has no separation violating condition (M2), The vertices u and v must lie in the set B. There are two simple cases now.

Case 1: u, $v \in A^* \cap B^*$ Then the order of (A^*, B^*) is exactly $|X|$. Moreover, $(G[B^*], A^* \cap B^*)$ is α-massed. Thus by the minimality of (G, X), (A^*, B^*) is a rigid separation, contrary to Lemma 2.1.

Case 2: Both u and v lie in $B^* - A^*$ We observed above that $(G/uv[B], A \cap B)$ contains a $\overline{\pi}$ rooted $\overline{H}$ minor for any subgraph $\overline{H}$ of H of size $|A \cap B|$. Given that u and v lie in $B^* - A^*$, then $A \cap B = A^* \cap B^*$. Then since $(G[B^*], A^* \cap B^*)$ contains a $\overline{\pi}$ rooted $\overline{H}$ minor,we see that $(G[B], A \cap B)$ does as well. We chose the map $\overline{\pi}$ and subgraph $\overline{H}$ arbitrarily, so the separation (A^*, B^*) is in fact a rigid separation, contradicting Lemma 2.1.

Thus contracting the edge uv must violate condition (M1), and the contraction of uv must have removed at least $\lfloor\alpha\rfloor + 1$ edges from G. Those edges either arise as common neighbors of u and v in G, or as edges that originally had only one end in the set X, and after contracting uv, now have two ends in X. The claim now follows.

$\square$

Claim 4.2 *There exists a vertex v in $V(G) - X$ such that* $\deg(v) \leq 2\alpha$.

Proof. By the definition of minimality, we know for any edge $e \not\subseteq X$ that $(G-e, X)$ is not α-minimal. Then $(G-e, X)$ must not be α-massed, implying that either $(G-e, X)$ fails to satisfy (M1) or (M2) in the definition of α-massed. Let $e = uv$ and assume there exists a separation (A, B) violating (M2). To prevent such a separation from violating (M2) in (G, X), we see that, without loss of generality, $u \in A - B$ and $v \in B - A$. By Claim 4.1, v has $\lfloor\alpha\rfloor$ neighbors that are either common neighbors with u or elements of X. In either case, v has $\lfloor\alpha\rfloor$ neighbors in A (other than the vertex v), implying that the order of (A, B) must be at least $\lfloor\alpha\rfloor$. But this contradicts our choice of α to be at least $|V(H)|$.

Thus we see that $(G-e, X)$ fails to satisfy (M1). This implies that $\rho(G-e, X) = \lfloor\alpha|V(G)-X|\rfloor + 1$. For every vertex $x \in X$, let $d^*(x)$ be the number of neighbors of x in $V(G) - X$. Then

$$2\rho(G - X) = \sum_{x \in X} d^*(x) + \sum_{v \in V(G)-X} \deg(v)$$

Every vertex $x \in X$ must have at least one neighbor y in $V(G) - X$, and by Claim 4.1, then x must have at least two neighbors in $|V(G) - X|$. Thus if the claim were false and $\deg(v) \geq 2\alpha$ for every $v \in V(G) - X$, we see

$$2\lfloor\alpha|V(G) - X|\rfloor + 2 \geq 2|X| + 2\alpha|V(G) - X|$$

which is false since we may assume $|X| \geq 3$. $\square$

$\square$

Proof. (Theorem 2.3)

Let G, X, H, H', and π be as in the statement of the theorem. Assume G does contain such an H universal subgraph G'. If G contained $|X|$ disjoint paths from X to the subgraph G', then clearly (G, X) would contain a π-rooted H' minor. Thus there exists a separation (A, B) of G such that $X \subseteq A$, $G' \subseteq B$, and the order of (A, B) is strictly less than $|X|$. But then such a separation chosen of minimal order will be a rigid separation, contrary to Theorem 2.1. $\square$

5 Proof of Theorem 1.1

We will prove the stronger statement:

Theorem 5.1 *Let G and H be graphs, and let $X \subseteq V(G)$ with $|X| \leq |V(H)|$. Let $t = |V(H)|$ and $c > 1$ be a real number such that every graph on n vertices with cn edges contains H as a minor. Let $\epsilon \geq 0$ be a real number, and let $f(\epsilon)$ be a function satisfying*

$$f(\epsilon) \geq \frac{12(6 + 2\sqrt{2} + \epsilon)(6 + 2\sqrt{2})(2 + 2\sqrt{2})}{\epsilon}$$

and

$$f(\epsilon) \geq 2\epsilon + 12 + 4\sqrt{2} + \frac{4(6 + 2\sqrt{2})}{2 + 2\sqrt{2}}.$$

Then if (G, X) is $\big((6 + 2\sqrt{2} + \epsilon)c + f(\epsilon)t\big)$-massed, (G, X) contains a π-rooted H' minor for all subgraphs H' of H with $|V(H')| = |X|$ and for all bijections $\pi : X \to V(H')$.

Proof. Let H, t, and c be given as in the statement of the theorem. Assume the theorem is false, and pick (G, X) to be a $\left(H, ((6+2\sqrt{2}+\epsilon)c + f(\epsilon)t)\right)$-minimal pair. Let H' and π be as in the definition of $\left(H, ((6+2\sqrt{2}+\epsilon)c + f(\epsilon)t)\right)$ minimality. Notice that by minimality, we may assume $t \geq |X| \geq 3$.

Consider a vertex $v \in V(G) - X$ of minimum degree, and let D be the subgraph of G induced by $v \cup N_v$. Theorems 2.1, and 2.2 imply that D has $\delta(D) \geq (6+2\sqrt{2}+\epsilon)c+(f(\epsilon)-1)t$ and $|D| \leq (12+4\sqrt{2}+2\epsilon)c+2f(\epsilon)t+1$. We now show, contrary to Theorem 2.3, that D contains an H-universal subgraph.

First, we will utilize the following observation.

Observation 5.2 *Let G be a graph and $d \leq 1$ a positive real number. Then there exists a subgraph H of G with $|V(H)| \leq \lceil d|V(G)| \rceil + 1$ with $|E(H)| \geq d^2|E(G)|$.*

Proof. The claim follows from a simple probabilistic argument. Let $n = |V(G)|$. Choose a set X of $\lceil d|V(G)| \rceil + 1$ vertices of G uniformly at random. For a given edge $e \in E(G)$, the probability that e has both ends in X is

$$\frac{\binom{n-2}{\lceil dn \rceil - 1}}{\binom{n}{\lceil dn \rceil + 1}} = \frac{(\lceil dn \rceil + 1)\lceil dn \rceil}{n(n-1)} \geq d^2$$

The expected number of edges with both ends in X is at least $d^2|E(G)|$, and there exists a subgraph achieving the expectation as desired. $\square$

Claim 5.3 *D has a separation of order at most $4c + \frac{4\epsilon}{6+2\sqrt{2}} + \left(\frac{4f(\epsilon)-2}{6+2\sqrt{2}} + 2\right)t + 3$.*

Proof. Assume to reach a contradiction that D is at least $4c + \frac{4\epsilon}{6+2\sqrt{2}} + \left(\frac{4f(\epsilon)-2}{6+2\sqrt{2}} + 2\right)t + 3$ connected.

Fix a set $X \subseteq V(D)$ with $|X| = t$. We set $d = \frac{2}{6+2\sqrt{2}}$ and utilize Observation 5.2. We have a subgraph L of $D - X$ with

$$|V(L)| \leq \frac{2}{6+2\sqrt{2}}(n-t) + 2 \leq 4c + \frac{4\epsilon}{6+2\sqrt{2}}c + \frac{4f(\epsilon)-2}{6+2\sqrt{2}}t + 3$$

and at least

$$\begin{aligned}
|E(L)| &\geq (\delta(D) - t)/2(n-t)\left(\frac{2}{6+\sqrt{2}}\right)^2 \\
&\geq \frac{(6+2\sqrt{2}+\epsilon)c + (f(\epsilon)-2)t}{2}(n-t)\left(\frac{2}{6+\sqrt{2}}\right)^2
\end{aligned}$$

edges. Re-ordering the terms, we see that

$$\begin{aligned}
|E(L)| &\geq \frac{2(n-t)}{(6+2\sqrt{2})^2}c + \frac{2\epsilon c}{(6+2\sqrt{2})^2}(n-t) + \frac{2(f(\epsilon)-2)t}{6+2\sqrt{2}}(n-t) \\
&\geq \frac{2(n-t)}{6+2\sqrt{2}}c + 3c \\
&\geq c|V(L)|.
\end{aligned}$$

By the definition of c, L contains an H minor with branch sets $\{S_1, \ldots, S_t\}$. Now notice that every vertex of D has at least $\delta - |L| - t$ neighbors in $D - L - X$.

$$\begin{aligned}\delta - |L| - t &\geq (6 + 2\sqrt{2} + \epsilon)c + (f(\epsilon) - 1)t - \\ &- 4c - \frac{4\epsilon}{6+2\sqrt{2}}c - \frac{4f(\epsilon)-2}{6+2\sqrt{2}}t - 3 - t \\ &\geq \left[f(\epsilon) - 2 - \frac{4f(\epsilon)-2}{6+2\sqrt{2}} \right] t \\ &\geq 10t\end{aligned}$$

Clearly, we can then pick distinct vertices $v_1, \ldots, v_t \in V(D) - V(L) - X$ such that v_i is adjacent a vertex of S_i for all $i = 1, \ldots, t$. Also, by assumption, we know that $G - L$ is $2t$ connected. By Theorem 1.2, $G - L$ is t linked. Then for any bijective map $\phi : X \to V(H)$, there exists disjoint paths P_i where the ends of P_i are x_i and $v_{\phi(i)}$. We see that (D, X) has a ϕ rooted H minor with branch sets $\overline{S_i} = P_i \cup S_{\phi(i)}$. Since ϕ and X were chosen arbitrarily, D is H-universal, contrary to Theorem 2.3. $\square$

Let (A, B) be a separation of exactly order $\left\lfloor 4c + \frac{4\epsilon}{6+2\sqrt{2}}c + \left(\frac{4f(\epsilon)-2}{6+2\sqrt{2}} + 2\right)t + 3 \right\rfloor$. Without loss of generality, we may assume that $|A| \leq |B|$. We will roughly iterate the argument above to show that $D[A - B]$ is an H universal subgraph of G.

Let $Z := A \cap B$, and fix a set of t vertices $X \subseteq A - Z$. Let D' be the subgraph $D[A - Z - X]$, and let $n' := |V(D')|$, and $\delta' := \delta(D')$. First we see that $n' \leq \frac{n-|Z|}{2} - t \leq (4+2\sqrt{2})c + \left(\epsilon - \frac{2\epsilon}{6+2\sqrt{2}}\right)c + \left(f(\epsilon) - \frac{2f(\epsilon)}{6+2\sqrt{2}}\right)t$. Moreover, $\delta' \geq \delta(D) - |Z| - t \geq (2+2\sqrt{2})c + \left(\epsilon - \frac{4\epsilon}{6+2\sqrt{2}}\right)c + \left(\frac{(2+2\sqrt{2})f(\epsilon)}{6+2\sqrt{2}} - 4\right)t - 3$.

Fix

$$d' := \frac{2}{(2+2\sqrt{2})\left(1 + \frac{\epsilon}{6+2\sqrt{2}}\right)}.$$

Now applying Observation 5.2 to find a subgraph L' satisfying the conclusions of the Observation. We see that $|V(L')| \leq d'n' + 2$, and $|E(L')| \geq d'n'\delta'/2d'$. Then

$$\frac{\delta'}{2}d' \geq \frac{1}{(2+2\sqrt{2})\left(1 + \frac{\epsilon}{6+2\sqrt{2}}\right)} \left[(2+2\sqrt{2})c + \left(\epsilon - \frac{4\epsilon}{6+2\sqrt{2}}\right)c + \left(\frac{f(\epsilon)(2+2\sqrt{2})}{6+2\sqrt{2}} - 4\right)t - 3 \right].$$

By our choice of $f(\epsilon)$, we see that

$$\frac{\delta'}{2}d' \geq c + 2.$$

Since $d'n' \geq \delta'd' \geq c + 2$, we have that

$$n'\frac{\delta'}{2}d'^2 \geq d'n'(c+2) \geq (d'n'+2)c.$$

Thus we see that D' contains an H minor with branch sets $\{S'_1, \ldots, S'_t\}$. Now if we consider any two non-adjacent vertices of $D[A - B]$, we see that they have at least $2\delta(D[A-B]) - |A - B| - |L'|$ common neighbors in $A - Z - V(L)$. We will now show that

$$2\delta(D[A-B]) - |A-B| - |L'| \geq 2t. \tag{1}$$

This will suffice to complete the proof. Pick $v'_i \in S'_i$ and a bijective map $\phi : X' \to V(H)$. Then x'_i and $v'_{\phi(i)}$ are either adjacent or have at least t common neighbors in D'. We can then link x'_i to $v'_{\phi(i)}$ using these common neighbors to get a ϕ- rooted H minor in (D', X'). Again, ϕ and X' were chosen arbitrarily, so D' is a universal subgraph, a contradiction.

All that remains is to prove inequality 1 holds.

$$\begin{aligned}
2\delta(D[A-B]) - |A-B| - |L'| &\geq 2\delta(D[A-B]) - (d'+1)n' - 2 - t \\
&\geq (4+4\sqrt{2})c + \frac{4+4\sqrt{2}}{6+2\sqrt{2}}\epsilon c + \left(\frac{4+4\sqrt{2}}{6+2\sqrt{2}} f(\epsilon) - 6\right) t - 6 - \\
&- \left(1 + \frac{2}{(2+2\sqrt{2})\left(1+\frac{\epsilon}{6+2\sqrt{2}}\right)}\right)\left[(4+2\sqrt{2})c + \frac{4+2\sqrt{2}}{6+2\sqrt{2}}\epsilon c + \frac{4+2\sqrt{2}}{6+2\sqrt{2}} f(\epsilon) t\right] \\
&- 2 - t
\end{aligned}$$

Examining the terms containing c, we see they are positive and can be disregarded. Utilizing the fact that $t \geq 3$, it suffices to show that

$$\frac{2\sqrt{2}}{6+2\sqrt{2}} f(\epsilon)t - \frac{8+4\sqrt{2}}{(2+2\sqrt{2})(6+2\sqrt{2})} \frac{1}{1+\frac{\epsilon}{6+2\sqrt{2}}} f(\epsilon)t \geq 12t.$$

But this follows from our choice of $f(\epsilon)$, completing the proof.

□

Proof. (Theorem 1.1) Let G, H, and c be as in the statement of the Theorem, and let $t = |V(H)|$. First, we observe by setting $\epsilon = 13 - 2\sqrt{2}$ and $f(13-2\sqrt{2}) = 1236$ satisfy the conditions concerning $f(\epsilon)$ in the statement of Theorem 5.1. Consider an arbitrary set $X \subseteq V(G)$ with $|X| = |V(H)|$. The pair (G, X) must satisfy condition (M2) because G is t connected. Also, $\rho(G - X) \geq |E(G)| - \binom{t}{2} \geq (18c + 1236t)(|V(G) - X|) + (18c + 1236t)t - t^2 \geq (18c + 1236t)(|V(G) - X|) + 1$. Thus (G, X) satisfies condition (M1), and applying Theorem 5.1 completes the proof. □

We would like to make a brief observation on the constants obtained in Theorem 1.1. It is certainly possible that the theorem could be improved; the value of 1236 could almost definitely be lowered. However, the goal of Theorem 5.1 was to find the optimal constant possible in front of the c term, using the proof technique, and in doing so, to make the minimal possible assumptions about c. This was desirable since as we consider larger and larger graphs for our H, we see that c_H need not grow linearly as a function of $|V(H)|$.

6 Proof of Theorem 1.3

We will actually prove a slightly stronger statement.

Theorem 6.1 *Let G be a graph and $X \subseteq V(G)$ with $|X| = s \leq t$. Then if (G, X) is t-massed, G contains a $K_{2,s}(X)$.*

Proof. Assume the theorem is false, and let G and X be a counter-example on a minimal number of vertices, and, subject to that, with $\rho(V(G) - X)$ minimized. Also, we may assume that $G[X]$ is a complete graph, since adding any edges to X will not affect the existence of a $K_{2,s}(X)$. We follow the proofs of Theorems 2.1 and 2.2. Since the structure $K_{2,s}(X)$ is not exactly a rooted minor according to our definition, we will in effect reprove Lemma 3.2 and Theorem 2.1 in Claims 6.2 and 6.3. Since the proofs are very similar, we will not go into extensive detail here.

A 2 - *bipartite rigid* separation (A, B) of (G, X) will be one where $X \subseteq A$ and $G[B]$ contains a $K_{2,|A\cap B|}(A \cap B)$.

Claim 6.2 *Let (A, B) be a 2-bipartite rigid separation of a pair (G, X) where $|X| = s$. Then if G' is the separation truncation of G, (G, X) contains a $K_{2,s}(X)$ if (G', X) contains a $K_{2,s}(X)$.*

Proof. Let $U_1, \dots, U_s, W_1, W_2$ be the branch sets of a $K_{2,s}(X)$ in (G', X). Define the graph $\overline{G}$ to be the graph with vertex set $A \cup \{z_1, z_2\}$ and edge set $E(G[A]) \cup \{z_1x, z_2x : x \in A \cap B\}$. In other words, $\overline{G}$ is the graph $G[A]$ with two additional vertices z_1 and z_2 adjacent all of the vertices in $A \cap B$. Clearly $\overline{G}$ is a minor of (G, X) and it suffices to prove that $(\overline{G}, X)$ contains a $K_{2,s}(X)$ minor.

For every $1 = 1, \dots, s$, let $\overline{U_i}$ be a maximal subset of U_i with the following properties:

1. $\overline{U_i} \cap X \neq \varnothing$
2. $\overline{U_i}$ induces a connected subgraph of $\overline{G}$, and
3. $\overline{U_i}$ contains at most one vertex of $A \cap B$.

Note that if U_i does intersect $A \cap B$, then by maximality, $\overline{U_i}$ contains at least one vertex of $A \cap B$. Also, by definition, the $\overline{U_i}$'s are connected and each contains exactly one vertex of X.

If W_i intersects $A \cap B$ for either $i = 1$ or 2, let $\overline{W_i} = A \cap W_i \cup z_i$. In this case, $\overline{W_i}$ induces a connected subgraph of $\overline{G}$ and has an edge going to every $\overline{U_i}$. Thus if both W_1 and W_2 intersect $A \cap B$, we have found a $K_{2,s}(X)$ in $\overline{G}$. If exactly one W_i, say W_1, intersects $A \cap B$, then consider W_2. W_2 induces a connected subgraph in $\overline{G}$ since it cannot use any edges of $A \cap B$. If W_2 has an edge to every $\overline{U_i}$, then $\{\overline{U_1}, \dots, \overline{U_s}, \overline{W_1}, W_2\}$ form a $K_{2,s}(X)$ in $\overline{G}$. Instead assume there is some index, say j, such that W_2 has no edge to $\overline{U_j}$. We know there is an edge in G' between the set W_2 and U_j. Let xy be such an edge, and let x be the end in U_j. Since W_2 does not intersect $A \cap B$, the edge xy is also present in $\overline{G}$. Look at a path in $G'[U_j]$ from x to $\overline{U_j}$. Such a path clearly exists since U_j induces a connected subgraph in G'. Moreover, by the maximality of the $\overline{U_j}$, such a path must intersect $A \cap B$ before reaching $\overline{U_j}$. Let P be a path from x to $A \cap B$ in $\overline{G}[U_j]$ that does not intersect $\overline{U_j}$. Now $\{\overline{U_1}, \dots, \overline{U_s}, W_1 \cup z_1, W_2 \cup V(P) \cup z_2\}$ form a $K_{2,s}(X)$ minor in $\overline{G}$.

We may now assume neither W_i intersects the set $A \cap B$. Let $\mathcal{N}_i$ be the set of indices k such that $\overline{U_k}$ has no edge in $\overline{G}$ to W_i. If both $\mathcal{N}_i$'s are empty, then clearly the sets $\{\overline{U_1}, \dots \overline{U_s}, W_1, W_2\}$ form a $K_{2,s}(X)$ minor in $\overline{G}$. If exactly one $\mathcal{N}_i$, say $\mathcal{N}_1$ is empty, we construct a $K_{2,s}(X)$ as follows. Let i be an index in $\mathcal{N}_2$. Then as in the previous paragraph, let P be a path in $\overline{G}[U_i]$ from W_2 to $A \cap B$. Then $\{\overline{U_1} \dots, \overline{U_s}, W_1, W_2 \cup V(P) \cup z_2\}$ form a $K_{2,s}(X)$ minor in $\overline{G}$.

We have now shown that both $\mathcal{N}_i$'s are non-empty. There are two distinct cases:

Case 1: There exists distinct representatives from $\mathcal{N}_1$ and $\mathcal{N}_2$ Let the distinct representatives be $j \in \mathcal{N}_1$ and $k \in \mathcal{N}_2$. As in the previous paragraph, let P_j be a path from W_1 to $A \cap B$ in $\overline{G}[U_j]$ disjoint from $\overline{U_j}$, and similarly define P_k. Then we get a $K_{2,s}(X)$ minor in $\overline{G}$ with branch sets $\{\overline{U_1}, \dots, \overline{U_s}, W_1 \cup P_j \cup z_1, W_2 \cup P_k \cup z_2\}$.

Case 2: No such distinct representatives from $\mathcal{N}_1$ and $\mathcal{N}_2$ exist. In this case, $\mathcal{N}_1 = \mathcal{N}_2 = \{k\}$ for some index k. Now define $U_i^* = \overline{U_i}$ for $i \neq k$ and $U_k^* = U_k \cup z_1$. Then $\{U_1^* \dots, U_s^*, W_1, W_2\}$ form a $K_{2,s}(X)$ minor in $\overline{G}$.

This completes the analysis, proving the claim. $\square$

Notice that unlike in Lemma 3.2, Claim 6.2, the implication is only in one direction. In fact, the converse is not true. It is possible that a graph G contains a $K_{2,t}(X)$ for some set X, and yet the separation truncation of a 2-bipartite rigid separation does not.

Claim 6.3 *(G, X) has no 2 - bipartite rigid separation.*

Proof. The proof follows the proof of Theorem 2.1. Pick such a separation (A, B) to minimize $|A|$. If the separation is of order at least $|X|$, then either there exist $|X|$ disjoint paths from X to $A \cap B$, or there exists a separation of smaller order in $G[A]$ separating X from $A \cap B$. Such a separation of minimal order induces a 2 - bipartite rigid separation violating our choice of (A, B).

Now assuming that the separation (A, B) is of order strictly less than $|X|$, consider the separation truncation G' of (A, B). If G' is t-massed, then by the minimality of our counterexample, G' would have a $K_{2,s}(X)$. Claim 6.2 implies G would also contain a $K_{2,s}(X)$, a contradiction. Thus we may assume that (G', X) is not

t - massed. By the fact that we know that the order of (A, B) is at most $s-1 \leq |X|-1$, we know that (G', X) satisfies condition (M1). Thus we may assume there exists a separation (A', B') violating condition (M2). Choose such a separation to minimize $|B'|$. Then $(G'[B'], A' \cap B')$ is t-massed, and consequently, $G'[B']$ contains a $K_{2,|A' \cap B'|}(A' \cap B')$ minor. Since $G'[A \cap B]$ is a complete subgraph, $A \cap B$ must be a subset of either A' or B'. Since $(A \cup A', B')$ would be a separation of (G, X) violating condition (M2) in G, we know that $A \cap B$ is a subset of B'. But then (B', B) is a 2-bipartite rigid separation of $(G[B' \cup B], A' \cap B')$. By applying Claim 6.2 to this separation, we see that $(A', B' \cup B)$ is a 2-bipartite rigid separation violating our choice of (A, B) to minimize $|A|$. □

Claim 6.4 *Every edge e of G with at most one end in X is in at least t triangles.*

Proof. Attempt to contract the edge e. By minimality, if $(G/e, X)$ were t-massed, then G/e would contain a $K_{2,s}(X)$ implying the existence of a $K_{2,s}(X)$ minor in G. Instead, it must be the case that $(G/e, X)$ is not t-massed. Assume that $(G/e, X)$ contains a separation (A, B) violating (M2), and assume that we chose it to minimize $|B|$. Then $(G/e[B], A \cap B)$ is t-massed. By minimality, $G/e[B]$ contains a $K_{2,|A \cap B|}$ minor. The separation (A, B) induces a separation (A^*, B^*) in G by uncontracting the edge e. If the ends of e lie in $A^* - B^*$, the separation would violate condition (M2). There are now two cases. First, consider when both the endpoints of e are contained in $A^* \cap B^*$. Then $\rho(B^* - A^*) \geq \rho(B - A)$ and $|B^* - A^*| = |B - A|$. Moreover $|A^* \cap B^*| \leq |A \cap B| + 1 \leq |X|$ and by minimality, (A^*, B^*) is a 2-bipartite rigid separation giving us a contradiction. However, in the other case, both ends of e lie in $B^* - A^*$. Then (A^*, B^*) is rigid since $G/e[B]$ contains an $K_{2,|A \cap B|}(A \cap B)$, again a contradiction.

Contracting the edge e must violate condition (M1). But because $G[X]$ is a complete subgraph, the edge e must lie in t triangles, as claimed. □

Then any edge e not contained in X has a $K_{2,t}$ subgraph using the common neighbors of its endpoints. We will now see that it is possible to find disjoint paths from X to one such $K_{2,t}$ subgraph and have all the paths avoid the bipartition of size 2.

Pick a separation (A, B) of order $|X|$ with $X \subseteq A$ chosen to minimize the $|B|$. If no such non-trivial separation exists, choose the trivial separation $(X, V(G))$. Now choose a separation (A', B') of $G[B]$ of order $|X| + 1$ with $A \cap B \subseteq A'$. Moreover, choose (A', B') to minimize $|B'|$. Again, if no such separation exists, choose pick an arbitrary vertex x of $B - A$ and use the trivial separation $(A \cap B \cup x, B)$. Notice that by Claim 6.4, x has degree at least $t + 1$, so there must exist some neighbor of x in $B - A$.

Let x be a vertex of $(A' \cap B') - (A \cap B)$ and y a neighbor of x in $B' - A'$. Let N be t common neighbors of x and y. In $G[B' - x - y]$, there exist $|X|$ disjoint paths from $A' \cap B' - x$ to N. Otherwise there would exist a separation of order $|X| - 1$, and adding x and y back in we would get a separation violating our choice of (A', B') to minimize $|B'|$. If we now look at $G[A']$, there exist $|X|$ disjoint paths from $A \cap B$ to $A' \cap B' - x$, or we would get a separation violating our choice of (A, B) to minimize $|B|$. However, now we have found $|X|$ disjoint paths from $A \cap B$ to N avoiding x and y. Thus (A, B) is a 2 - bipartite rigid separation, contrary to Claim 6.3. This completes the proof.

□

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