Relation between pairs of representations of signed binary matroids

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Abstract

We show how pairs of signed graphs with the same even cycles relate to pairs of grafts with the same even cuts. These results are proved in the more general context of signed binary matroids.

1 Introduction

We assume that the reader is familiar with the basics of matroid theory. See Oxley [4] for the definition of the terms used here. We will only consider binary matroids in this paper. Thus the reader should substitute the term “binary matroid” every time “matroid” appears in this text.
Given a graph $G$ and $X \subseteq E(G)$, $G[X]$ denotes the graph induced by $X$. A subset $C$ of edges is a cycle if $G[C]$ is a graph where every vertex has even degree. An inclusion-wise minimal non-empty cycle is a circuit. We denote by cycle$(G)$ the set of all cycles of $G$. Since the cycles of $G$ correspond to the cycles of the graphic matroid represented by $G$, we identify cycle$(G)$ with that matroid. Given two graphs $G_1$ and $G_2$ such that cycle$(G_1) = \text{cycle}(G_2)$, what is the relation between $G_1$ and $G_2$? This is answered by the following theorem of Whitney [8].

**Theorem 1.** For a pair of graphs $G_1$ and $G_2$, cycle$(G_1) = \text{cycle}(G_2)$ if and only if $G_1$ and $G_2$ are related by a sequence of Whitney-flips.

We need to define the term “Whitney-flip”. Consider a graph $G$ and a partition $X, \bar{X}$ of $E(G)$ where $|X|, |\bar{X}| \geq 2$ and $V(G[X]) \cap V(G[\bar{X}]) = \{u_1, u_2\}$, for some $u_1, u_2 \in V(G)$. Let $G'$ be obtained by identifying vertices $u_1, u_2$ of $G[X]$ with vertices $u_2, u_1$ of $G[\bar{X}]$ respectively. Then $G$ and $G'$ are related by a Whitney-flip. A Whitney-flip will also be the operation which consists of identifying two vertices from distinct components, as well as the operation consisting of partitioning the graph into components each of which is a block of $G$. An example of two graphs related by a Whitney-flip is given in Figure 1. In this example the set $X$ is given by edges $5, 6, 9, 10$. As Whitney-flips preserve cycles, sufficiency of the previous theorem is immediate.

![Figure 1: Example of a Whitney-flip.](image)

Given a graph $G$, we denote by cut$(G)$ the set of all cuts of $G$. Since the cuts of $G$ correspond
to the cycles of the cographic matroid represented by $G$, we identify $\text{cut}(G)$ with that matroid. Since the matroids $\text{cut}(G)$ and $\text{cycle}(G)$ are duals of one another, we can reformulate Theorem 1 as follows.

**Theorem 2.** For a pair of graphs $G_1$ and $G_2$, $\text{cut}(G_1) = \text{cut}(G_2)$ if and only if $G_1$ and $G_2$ are related by a sequence of Whitney-flips.

A signed graph is a pair $(G, \Sigma)$ where $\Sigma \subseteq E(G)$. We call $\Sigma$ a signature of $G$. A subset $D \subseteq E(G)$ is $\Sigma$-even (resp. $\Sigma$-odd) if $|D \cap \Sigma|$ is even (resp. odd). When there is no ambiguity we omit the prefix $\Sigma$ when referring to $\Sigma$-even and $\Sigma$-odd sets. In particular we refer to odd and even edges and cycles. We denote by $\text{ecycle}(G, \Sigma)$ the set of all even cycles of $(G, \Sigma)$. It can be verified that $\text{ecycle}(G, \Sigma)$ is the set of cycles of a binary matroid (with ground set $E(G)$) which we call the even cycle matroid represented by $(G, \Sigma)$. We identify $\text{ecycle}(G, \Sigma)$ with that matroid. Signed graphs are a special case of biased graphs [9] and even cycle matroids are a special case of lift matroids [10].

A graft is a pair $(G, T)$ where $G$ is a graph, $T \subseteq V(G)$ and $|T|$ is even. A cut $\delta_G(U) := \{uv \in E(G) : u \in U, v \notin U \}$ is $T$-even (resp. $T$-odd) if $|U \cap T|$ is even (resp. odd). When there is no ambiguity we omit the prefix $T$ when referring to $T$-even and $T$-odd cuts. We denote by $\text{ecut}(G, T)$ the set of all even cuts of $(G, T)$. It can be verified that $\text{ecut}(G, T)$ is the set of cycles of a binary matroid (with ground set $E(G)$) which we call the even cut matroid represented by $(G, T)$. We identify $\text{ecut}(G, T)$ with that matroid.

**Question 1:** Let $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ be signed graphs.

When is the relation $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$ satisfied?

**Question 2:** Let $(G_1, T_1)$ and $(G_2, T_2)$ be grafts.
When is the relation \( \text{cut}(G_1, T_1) = \text{cut}(G_2, T_2) \) satisfied?

Theorem 1 answers Question 1 for the case where \( \Sigma_1 = \Sigma_2 = \emptyset \) and Theorem 2 answers Question 2 for the case where \( T_1 = T_2 = \emptyset \). Slilaty [6] addresses a special case of the corresponding problem for biased matroids. The corresponding problem for bicircular matroids has been widely studied and a complete characterization of when two graphs represent the same bicircular matroid is known (see [7], [1] and [3]).

We know that Theorems 1 and 2 are equivalent. In an upcoming paper we shall provide a complete answer to Question 1 for three general classes of signed graphs.

The main result of this paper is Theorem 4, which illustrates the relation between Question 1 and Question 2. We introduce and explain this result in Section 2. In Section 3 we generalize to matroids the concepts introduced in the previous section and we state and prove more general results about signed binary matroids. Section 3.2 shows how the results on signed matroids imply the results for signed graphs and grafts; this section also provides a more detailed version of Theorem 4 (in Proposition 17). Section 3.3 introduces a matroid construction that leads to matroid-preserving operations on graphs.

## 2 Even cycles and even cuts

### 2.1 Matrix representations

Even cycle and even cut matroids are binary matroids: we now explain how to obtain their matrix representation from a signed graph or a graft representation.

Let \((G, \Sigma)\) be a signed graph. Let \(A(G)\) be the incidence matrix of \(G\), i.e. the columns of
$A(G)$ are indexed by the edges of $G$, the rows of $A(G)$ are indexed by the vertices of $G$ and entry $(v,e)$ of $A(G)$ is 1 if vertex $v$ is incident to edge $e$ in $G$ and 0 otherwise. Then $A(G)$ is a matrix representation of the graphic matroid represented by $G$. We obtain a matrix representation of ecycle($G,\Sigma$) as follows. Let $S$ be the transpose of the characteristic vector of $\Sigma$; hence $S$ is a row vector indexed by $E(G)$ and $S_e$ is 1 if $e \in \Sigma$ and 0 otherwise. Let $A$ be the binary matrix obtained from $A(G)$ by adding row $S$. Let $M(A)$ be the binary matroid represented by $A$. Every cycle $C$ of $M(A)$ corresponds to a cycle of $G$; moreover, $C$ intersects $\Sigma$ with even parity. Thus $M(A) = \text{ecycle}(G,\Sigma)$. Suppose we add an odd loop $\Omega$ to $(G,\Sigma)$. This corresponds to adding a new column to $A$ which has all zero entries, except for row $S$, which contains a one. Hence contracting $\Omega$ in the new matrix we obtain the matrix $A(G)$. It follows that every even cycle matroid is a lift of a graphic matroid. Note that in constructing $A$ we may replace $A(G)$ with any binary matrix whose rows span the cut space of $G$.

Now we describe how to obtain a matrix representation of an even cut matroid from a graft representation. For a graph $G$, a set $J \subseteq E(G)$ is a $T$-join if $T$ is the set of vertices of odd degree of $G[J]$. Let $(G,T)$ be a graft and $J$ a $T$-join of $G$. Let $\hat{A}(G)$ be a binary matrix whose rows span the cycle space of $G$. Hence $\hat{A}(G)$ is a matrix representation of the cographic matroid represented by $G$. Let $\hat{S}$ be the transpose of the incidence vector of $J$, that is, $\hat{S}$ is a row vector indexed by $E(G)$ and $\hat{S}_e$ is 1 if $e \in J$ and 0 otherwise. Construct a matrix $\hat{A}$ from $\hat{A}(G)$ by adding row $\hat{S}$. Let $M(\hat{A})$ be the binary matroid represented by $\hat{A}$. Let $C$ be a cycle of $M(\hat{A})$; then $C$ corresponds to a cut of $G$ and intersects $J$ with even parity. It is easy to see that a cut intersects $J$ with even parity if and only if it is $T$-even. Thus $M(\hat{A}) = \text{ecut}(G,T)$. Suppose we uncontract an odd bridge $\Omega$ in $(G,\Sigma)$. This corresponds to adding a new column to $\hat{A}$ which has all zero entries, except for row $S$, which contains a one; this is because $\Omega$ is not contained in any cycle.
of \(G\) and \(\Omega\) is contained in every \(T\)-join of \(G\). Hence contracting \(\Omega\) in the new matrix we obtain the matrix \(\hat{A}(G)\). It follows that every even cut matroid is a lift of a cographic matroid.

### 2.2 Isomorphism

The following theorem of Gerards, Lovász, Schrijver, Seymour, Shih and Truemper (see [2]) answers Question 1 for the class of even cycle matroids within the class of graphic matroids.

**Theorem 3.** Let \((G, \Sigma)\) and \((G', \Sigma')\) be signed graphs. Suppose that \(\text{ecycle}(G, \Sigma) = \text{ecycle}(G', \Sigma')\) and that this matroid is graphic. Then \((G, \Sigma)\) and \((G', \Sigma')\) are related by a sequence of Whitney-flips, signature exchanges, and Lovász-flips.

We need to define the terms “signature exchange” and “Lovász-flip”. A set \(\Sigma' \subseteq E(G)\) is a signature of \((G, \Sigma)\) if \(\text{ecycle}(G, \Sigma) = \text{ecycle}(G, \Sigma')\). It can be readily checked that \(\Sigma'\) is a signature of \((G, \Sigma)\) if and only if \(\Sigma' = \Sigma \Delta D\) for some cut \(D\) of \(G\). The operation that consists of replacing a signature of a signed graph by another signature is a signature exchange.

Given a graph \(G\) we denote with \(\text{loop}(G)\) the set of loops of \(G\). Consider a signed graph \((G, \Sigma)\) and vertices \(v_1, v_2 \in V(G)\) where \(\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2) \cup \text{loop}(G)\). We can construct a signed graph \((G', \Sigma)\) from \((G, \Sigma)\) by replacing the adjacencies of every odd edge \(e\) as follows:

- if \(e = v_1v_2\) in \(G\), then \(e\) becomes a loop in \(G'\) incident to \(v_1\);
- if \(e\) is a loop in \(G\), then \(e = v_1v_2\) in \(G'\);
- if \(e = xv_i\), for \(1 \leq i \leq 2\) and \(x \neq v_1, v_2\), then \(e = xv_{3-i}\) in \(G'\).

In this case, we say that \((G', \Sigma)\) is obtained from \((G, \Sigma)\) by a Lovász-flip. An example of two signed graphs related by a Lovász-flip is given in Figure 2, where the white vertices represent
the vertices $v_1, v_2$ in the definition. Note that Lovász-flips are only well defined on signed graphs $(G, \Sigma)$ where $\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2) \cup \text{loop}(G)$ for some vertices $v_1$ and $v_2$. It can be easily verified that Lovász-flips preserve even cycles.

![Signed graphs related by a Lovász-flip. Bold edges are odd.](image)

An interesting result of Shih (see [5]) characterizes the relation between graphs $G$ and $H$ such that the cycle space of $H$ is a codimension-1 subspace of the cycle space of $G$. In particular, this characterizes when $\text{cycle}(H) = \text{ecycle}(G, \Sigma)$ for a graph $H$ and a signed graph $(G, \Sigma)$. In fact, if $\text{cycle}(H)$ and $\text{cycle}(G)$ have the same dimension, then by Theorem 1 they are related by Whitney-flips. If they have different dimensions, then $\text{cycle}(H)$ is a codimension-1 subspace of $\text{cycle}(G)$, because, for a fixed odd cycle $C$ of $(G, \Sigma)$, every other odd cycle $D$ of $G$ can be written as the symmetric difference of $C$ and the even cycle $C \triangle D$.

### 2.3 Equivalence of Question 1 and 2

Suppose that $\text{cycle}(G_1) = \text{cycle}(G_2)$ for graphs $G_1$ and $G_2$. Then $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$ if and only if $\Sigma_1 \triangle \Sigma_2$ is a cut of $G_1$ (equivalently of $G_2$). Moreover, $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$ if and only if any $T_1$-join of $G_1$ is a $T_2$-join of $G_2$ (recall that, for a graph $H$, a set $J \subseteq E(H)$ is a $T$-join if $T$ is the set of vertices of odd degree of $H[J]$). Thus Questions 1 and 2 are triv-
ial for the case when $\text{cycle}(G_1) = \text{cycle}(G_2)$. A signed graph $(G, \Sigma)$ is \textit{bipartite} if $G$ has no $\Sigma$-odd cycle. The following theorem shows the relation between the two questions in the case $\text{cycle}(G_1) \neq \text{cycle}(G_2)$.

\textbf{Theorem 4.} Let $G_1$ and $G_2$ be graphs such that $\text{cycle}(G_1) \neq \text{cycle}(G_2)$.

(1) Suppose there exists a pair $\Sigma_1, \Sigma_2 \subseteq E(G_1)$ such that $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$. For $i = 1, 2$, if $(G_i, \Sigma_i)$ is bipartite define $C_i := \emptyset$, otherwise let $C_i$ be a $\Sigma_i$-odd cycle of $G_i$. Let $T_{3-i}$ be the vertices of odd degree in $G_{3-i}[C_i]$. Then $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$.

(2) Suppose there exists a pair $T_1 \subseteq V(G_1)$ and $T_2 \subseteq V(G_2)$ (where $|T_1|$ and $|T_2|$ are even) such that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. For $i = 1, 2$, if $T_i = \emptyset$ let $\Sigma_{3-i} = \emptyset$, otherwise let $t_i \in T_i$ and let $\Sigma_{3-i} := \delta_{G_i}(t_i)$. Then $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$.

The proof of Theorem 4 is postponed until Section 3.2. We illustrate this result with an example. Consider the signed graphs $(G_i, \Sigma_i)$, for $i = 1, 2, 3$, in Figure 3. $(G_2, \Sigma_2)$ is obtained from $(G_1, \Sigma_1)$ by a Lovász-flip on vertices $b, f$; $(G_3, \Sigma_3)$ is obtained from $(G_2, \Sigma_2)$ by first a signature exchange $\Sigma_3 := \Sigma_2 \Delta \delta_{G_2}(b)$, then by performing a Lovász-flip on vertices $a, f$. As Lovász-flips and signature exchanges preserve even cycles, $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_3, \Sigma_3)$. In the same

![Figure 3: Bold edges of $G_i$ are in $\Sigma_i$, square vertices of $G_1, G_3$ are $T_1, T_3$.](image-url)
figure consider the grafts \((G_1, T_1)\) and \((G_3, T_3)\) where \(T_1 = \{a, b\}\) and \(T_3 = \{b, f\}\). These grafts are obtained using the construction in Theorem 4(1). If we consider the odd cycle \(\{4, 7, 9\}\) of \((G_1, \Sigma_1)\), then \(T_3\) is the set of vertices of odd degree in \(G_3[\{4, 7, 9\}]\). Similarly, if we consider the odd cycle \(\{1, 8, 9\}\) of \((G_3, \Sigma_3)\), then \(T_1\) is the set of vertices of odd degree in \(G_1[\{1, 8, 9\}]\). The theorem states that \(\text{ecut}(G_1, T_1) = \text{ecut}(G_3, T_3)\). We can also consider the reverse construction as in Theorem 4(2). Pick \(a \in T_1\), then \(\delta_{G_1}(a) = \{1, 2\}\). Now \(\{1, 2\} \triangle \Sigma_3\) is a cut of \(G_3\), hence \(\{1, 2\}\) is a signature of \((G_3, \Sigma_3)\). Similarly, pick \(b \in T_3\), then \(\delta_{G_3}(b) = \{1, 6, 7, 8\}\) is a signature of \((G_1, \Sigma_1)\).

The following results will follow immediately from Proposition 11 and Remark 16.

**Remark 5.** Given graphs \(G_1\) and \(G_2\) with cycle\((G_1) \neq \text{cycle}(G_2)\), there exists, up to signature exchanges, at most one pair \(\Sigma_1, \Sigma_2\) such that \(\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)\) and there exists at most one pair \(T_1, T_2\) such that \(\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)\).

The condition that \(\text{cycle}(G_1) \neq \text{cycle}(G_2)\) is necessary for uniqueness, as otherwise any pair \(\Sigma_1 = \Sigma_2\) will yield the same even cycles.

Suppose we can answer Question 1, does Theorem 4 then provide us with an answer to Question 2? Consider grafts \((G, T)\) and \((G', T')\), where \(\text{ecut}(G, T) = \text{ecut}(G', T')\). By Theorem 4 there exist signatures \(\Sigma\) and \(\Sigma'\) such that \(\text{ecycle}(G, \Sigma) = \text{ecycle}(G', \Sigma')\). Suppose we know a sequence of signed graphs \((G_i, \Sigma_i)\), for \(i \in [n]\), with \((G, \Sigma) = (G_1, \Sigma_1)\) and \((G', \Sigma') = (G_n, \Sigma_n)\), where \((G_i, \Sigma_i)\) have the same even cycles for all \(i \in [n]\). (For example, this is the case when \(T' = \emptyset\) (i.e. \(\text{ecut}(G', T')\) is cographic) as then \(\Sigma = \emptyset\) (i.e. \(\text{ecycle}(G, \Sigma)\) is graphic) and Theorem 3 then describes how \((G, \Sigma)\) and \((G', \Sigma')\) are related.) Can we find sets \(T_i\), for all \(i \in [n]\), such that \(T = T_1, T' = T_n\) where \((G_i, T_i)\) have the same even cuts for all \(i \in [n]\)? The example in Figure 3
shows that this is not always the case. Remark 5 states that graphs $G_1$ and $G_3$ determine $T_1$ and $T_3$ uniquely. However, it is not possible to find a set $T_2$ such that $ecut(G_1, T_1) = ecut(G_2, T_2)$, because the edge 9 is a loop in $G_2$ but is contained in the $T_1$-even cut $\{6, 7, 8, 9\}$ of $G_1$.

This leads to the following definition: a set of graphs $\{G_1, \ldots, G_n\}$ is harmonious if, for all distinct $i, j \in [n]$, cycle($G_i$) $\neq$ cycle($G_j$) and there exist $\Sigma_1, \ldots, \Sigma_n$ and $T_1, \ldots, T_n$ such that $ecycle(G_i, \Sigma_i) = ecycle(G_j, \Sigma_j)$ and $ecut(G_i, T_i) = ecut(G_j, T_j)$, for all $i, j \in [n]$. For instance, the set $\{G_1, G_2, G_3\}$ in Figure 3 is not harmonious. In fact no large set of graphs is harmonious.

**Theorem 6.** Suppose that $\{G_1, \ldots, G_n\}$ is a harmonious set of graphs. Then $n \leq 3$.

Note that, in contrast, it is easy to construct arbitrarily large sets of graphs $\{G_1, \ldots, G_n\}$ such that, for all distinct $i, j \in [n]$, cycle($G_i$) $\neq$ cycle($G_j$) and for which there exist $\Sigma_1, \ldots, \Sigma_n$ such that $ecycle(G_i, \Sigma_i) = ecycle(G_j, \Sigma_j)$, for all $i, j \in [n]$.

The bound of 3 is best possible. A construction that yields a harmonious set of 3 graphs $\{G_1, G_2, G_3\}$ is as follows: let $(G_1, \Sigma_1)$ be any signed graph with vertices $u, v$ where $\Sigma_1 \subseteq \delta_{G_1}(u) \cup \delta_{G_1}(v)$. Let $(G_2, \Sigma_2)$ be obtained from $(G_1, \Sigma_1)$ by a Lovász-flip on $u, v$, and let $(G_3, \Sigma_3)$ be obtained from $(G_1, \Sigma_1 \triangle \delta_{G_1}(u))$ by a Lovász-flip on $u, v$. Finally, let $T_1 = \{u, v\}$ and for $i = 2, 3$, let $T_i$ be the vertices in $G_i$ corresponding to $u, v$.

The proof of Theorem 6 follows immediately from the proof of Theorem 15 in Section 3.1.4.

### 3 Generalization to signed matroids

In this section we will generalize to matroids the concepts introduced in the previous section.

Recall that all the matroids considered in this work are binary. Let $M$ be a matroid and let $\Sigma \subseteq E(M)$. A cycle of $M$ is any subset $C \subseteq E(M)$ such that $C$ is equal to the union of pairwise
disjoint circuits. A **cocycle** is a cycle of the dual $M^*$ of $M$. A pair $(M, \Sigma)$, where $\Sigma \subseteq E(M)$, is a **signed matroid**. A subset $D \subseteq E(M)$ is $\Sigma$-**even** (resp. $\Sigma$-$\text{odd}$) if $|D \cap \Sigma|$ is even (resp. odd).

The set of all cycles of $M$ that are $\Sigma$-even forms the set of cycles of a matroid which we denote by $\text{ecycle}(M, \Sigma)$. We say that $\Sigma'$ is a **signature** of $(M, \Sigma)$ if $\text{ecycle}(M, \Sigma) = \text{ecycle}(M, \Sigma')$. It can be readily checked that $\Sigma'$ is a signature of $(M, \Sigma)$ if and only if $\Sigma' = \Sigma \triangle D$ for some cocycle $D$ of $M$. The operation that consists of replacing a signature of a signed matroid by another signature is called **signature exchange**. A signed matroid $(M, \Sigma)$ is **bipartite** if all cycles of $M$ are $\Sigma$-even. It can be easily verified that $(M, \Sigma)$ is bipartite if and only if $\Sigma$ is a cocycle of $M$.

When $M = \text{cycle}(G)$ for some graph $G$, then $\text{ecycle}(M, \Sigma) = \text{ecycle}(G, \Sigma)$ and the reader should verify that in this context the aforementioned definitions for signed matroids correspond to the definitions for signed graphs.

### 3.1 Results for signed matroids

#### 3.1.1 Pairs

Let $(M_1, \Sigma_1)$ and $(M_2, \Sigma_2)$ be signed matroids such that $\text{ecycle}(M_1, \Sigma_1) = \text{ecycle}(M_2, \Sigma_2)$. A cycle (resp. cocycle) $C$ of $M_1$ is **preserved** from $M_1$ to $M_2$ if $C$ is also a cycle (resp. cocycle) of $M_2$. A signature of $(M_1, \Sigma_1)$ is **preserved** if it is a signature of $(M_2, \Sigma_2)$. Isomorphism problems for signed binary matroids arise in pairs.

**Theorem 7.** Let $(M_1, \Sigma_1)$ and $(M_2, \Sigma_2)$ be signed matroids with $\text{ecycle}(M_1, \Sigma_1) = \text{ecycle}(M_2, \Sigma_2)$. Then there exist $\Gamma_1, \Gamma_2 \subseteq E(M_1)$ such that $\text{ecycle}(M_1^*, \Gamma_1) = \text{ecycle}(M_2^*, \Gamma_2)$ and, for $i = 1, 2$, the $\Gamma_i$-even cocycles of $M_i$ are exactly the preserved cocycles of $M_i$. Moreover, if $(M_i, \Sigma_i)$ is bipartite, then so is $(M_{3-i}^*, \Gamma_{3-i})$. 

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The proof requires a number of preliminaries and is postponed to the end of the section. Given a signed matroid \((M, \Sigma)\), the cocycles of \(\text{ecycle}(M, \Sigma)\) are the sets that intersect every \(\Sigma\)-even cycle of \(M\) with even cardinality. Thus we have the following.

**Remark 8.** The cocycles of \(\text{ecycle}(M, \Sigma)\) are the cocycles of \(M\) and the signatures of \((M, \Sigma)\), which in turn implies the following.

**Remark 9.** Suppose that \(\text{ecycle}(M_1, \Sigma_1) = \text{ecycle}(M_2, \Sigma_2)\).

1. If \(D\) is a non-preserved cocycle of \(M_1\) then \(D\) is a signature of \((M_2, \Sigma_2)\).
2. If \(D\) is a non-preserved signature of \((M_1, \Sigma_1)\) then \(D\) is a cocycle of \(M_2\).

**Proof.** For both (1) and (2), Remark 8 implies that \(D\) is a cocycle of \(\text{ecycle}(M_1, \Sigma_1)\) and hence of \(\text{ecycle}(M_2, \Sigma_2)\). Remark 8 implies that \(D\) is either a cocycle of \(M_2\) or a signature of \((M_2, \Sigma_2)\).

For (1) \(D\) is not a cocycle of \(M_2\), for (2) \(D\) is not a signature of \((M_2, \Sigma_2)\). ☐

**Lemma 10.** Suppose that \(\text{ecycle}(M_1, \Sigma_1) = \text{ecycle}(M_2, \Sigma_2)\), for signed matroids \((M_1, \Sigma_1)\) and \((M_2, \Sigma_2)\). For \(i = 1, 2\), there exists \(\Gamma_i \subseteq E(M_i)\) such that, for every cocycle \(D\) of \(M_i\), \(D\) is preserved if and only if it is \(\Gamma_i\)-even. Moreover, if \((M_{3-i}, \Sigma_{3-i})\) is bipartite, then \(\Gamma_i = \emptyset\).

**Proof.** Fix \(i \in \{1, 2\}\). Let \(B\) be a cobasis of \(M_i\). For any \(e \notin B\), let \(D_e\) denote the unique cocircuit in \(B \cup \{e\}\) (these are the fundamental cocircuits of \(M_i\)). Then we let \(e \in \Gamma_i\) if and only if \(D_e\) is non-preserved. Consider now an arbitrary cocycle \(D\) of \(M_i\). \(D\) may be expressed as the symmetric difference of a set of distinct fundamental cocircuits \(D_e\), where, say, \(s\) of these are non-preserved. By construction, \(|D \cap \Gamma_i| = s\). By Remark 9(1), non-preserved cocycles of \(M_i\) are signatures of \((M_{3-i}, \Sigma_{3-i})\). Moreover, the symmetric difference of an even (resp. odd)
number of signatures of \((M_{3-i}, \Sigma_{3-i})\) is a cocycle of \(M_{3-i}\) (resp. a signature of \((M_{3-i}, \Sigma_{3-i})\)). It follows that \(D\) is a cocycle of \(M_{3-i}\) when \(s\) is even and is a signature of \((M_{3-i}, \Sigma_{3-i})\) when \(s\) is odd. If \((M_{3-i}, \Sigma_{3-i})\) is non-bipartite, then signatures of \((M_{3-i}, \Sigma_{3-i})\) are not cocycles of \(M_{3-i}\) and the result follows. If \((M_{3-i}, \Sigma_{3-i})\) is bipartite, then every cocycle of \(M_i\) is preserved. As a consequence, \(\Gamma_i = \emptyset\) and the result follows as well.

**Proof of Theorem 7.** Lemma 10 implies that, for \(i = 1, 2\), there exists \(\Gamma_i \subseteq E(M_1)\) such that the preserved cocycles of \(M_i\) are exactly the \(\Gamma_i\)-even cocycles of \(M_i\). Hence, \(\text{ecycle}(M_1^*, \Gamma_1) = \text{ecycle}(M_2^*, \Gamma_2)\). Again by Lemma 10, if \((M_i, \Sigma_i)\) is bipartite, then \(\Gamma_{3-i} = \emptyset\), so \((M_{3-i}^*, \Gamma_{3-i})\) is bipartite.

3.1.2 Uniqueness

The main observation in this section is the following.

**Proposition 11.** Suppose \((M_1, \Sigma_1)\) and \((M_2, \Sigma_2)\) are signed matroids such that \(M_1 \neq M_2\) and \(\text{ecycle}(M_1, \Sigma_1) = \text{ecycle}(M_2, \Sigma_2)\). For \(i = 1, 2\), the \(\Sigma_i\)-even cycles of \(M_i\) are exactly the preserved cycles of \(M_i\). In particular, \(\Sigma_1\) and \(\Sigma_2\) are unique up to signature exchanges.

Proposition 11 follows directly from the next remark.

**Remark 12.** Suppose \(\text{ecycle}(M_1, \Sigma_1) = \text{ecycle}(M_2, \Sigma_2)\). If \(C\) is a \(\Sigma_1\)-odd cycle of \(M_1\) which is preserved, then \(\text{cycle}(M_1) = \text{cycle}(M_2)\).

**Proof.** For any odd cycle \(D\) of \((M_1, \Sigma_1)\), let \(B := D \Delta C\). As \(B\) is an even cycle of \((M_1, \Sigma_1)\), we know that \(B\) is an even cycle of \((M_2, \Sigma_2)\), hence \(D\) is an odd cycle of \((M_2, \Sigma_2)\). Hence, \(\text{cycle}(M_1) \subseteq \text{cycle}(M_2)\). As \(C\) is a cycle of \(M_1\) and \(M_2\) and \(C\) is \(\Sigma_1\)-odd, \(C\) is also a preserved \(\Sigma_2\)-odd cycle of \(M_2\). Hence by symmetry the reverse inclusion holds as well.

\[\text{cycle}(M_1) = \text{cycle}(M_2)\]
3.1.3 Odd cycles and signatures

**Remark 13.** Suppose that $\text{e-cycle}(M_1, \Sigma_1) = \text{e-cycle}(M_2, \Sigma_2)$, for signed matroids $(M_1, \Sigma_1)$ and $(M_2, \Sigma_2)$, where $M_1 \neq M_2$. If $(M_1, \Sigma_1)$ is bipartite, then let $\Sigma := \emptyset$. Otherwise there exists a non-preserved cocycle $D$ of $M_2$; let $\Sigma := D$. Then $\Sigma$ is a signature of $(M_1, \Sigma_1)$.

**Proof.** We may assume that $(M_1, \Sigma_1)$ is non-bipartite. By Theorem 7, there exist $\Gamma_1$ and $\Gamma_2$ such that, for $i = 1, 2$, the $\Gamma_i$-even cocycles of $M_i$ are exactly the preserved cocycles of $M_i$. If every cocycle of $M_2$ is preserved, then $(M_2^*, \Gamma_2)$ is bipartite. It follows, from Theorem 7 applied to $(M_1^*, \Gamma_1)$ and $(M_2^*, \Gamma_2)$, and from Proposition 11, that $(M_1, \Sigma_1)$ is bipartite, a contradiction. Hence, some cocycle $D$ of $M_2$ is non-preserved. The result then follows by Remark 9(1).

The signature $\Sigma$ of $(M_1, \Sigma_1)$ in Remark 13 is called an $M_2$-standard signature.

**Theorem 14.** Let $(M_1, \Sigma_1)$ and $(M_2, \Sigma_2)$ be signed matroids such that $M_1 \neq M_2$ and let $\Gamma_1 \subseteq E(M_1)$ and $\Gamma_2 \subseteq E(M_2)$. Assume that $\text{e-cycle}(M_1, \Sigma_1) = \text{e-cycle}(M_2, \Sigma_2)$ and $\text{e-cycle}(M_1^*, \Gamma_1) = \text{e-cycle}(M_2^*, \Gamma_2)$. If, for $i = 1, 2$, $\Sigma_i$ is an $M_i$-$i$-standard signature, then for any $D \subseteq E(M_1)$ the following hold.

1. Suppose that $(M_1, \Sigma_1)$ is non-bipartite. Then
   
   $D$ is a $\Sigma_1$-odd cycle of $M_1$ if and only if $D$ is a $\Sigma_2$-even signature of $(M_2^*, \Gamma_2)$.

2. Suppose that $(M_1, \Sigma_1)$ and $(M_2, \Sigma_2)$ are non-bipartite. Then
   
   $D$ is a $\Sigma_1$-odd signature of $(M_1^*, \Gamma_1)$ if and only if $D$ is a $\Sigma_2$-odd signature of $(M_2^*, \Gamma_2)$.

**Proof.** We begin with the proof of (1). Let $D$ be a $\Sigma_1$-odd cycle of $M_1$. Remark 12 implies that $D$ is non-preserved. Remark 9(1) implies that $D$ is a signature of $(M_2^*, \Gamma_2)$. If $\Sigma_2 = \emptyset$, then $D$ is trivially $\Sigma_2$-even. Otherwise, as $\Sigma_2$ is a standard signature, $\Sigma_2$ is a cocycle of $M_1$. Since $M_1$ is a
binary matroid, cycles and cocycles have an even intersection, hence $D$ is $\Sigma_2$-even. Conversely, let $D$ be a $\Sigma_2$-even signature of $(M_2^*, \Gamma_2)$. As $(M_1, \Sigma_1)$ is non-bipartite, there exists a $\Sigma_1$-odd cycle $C$ of $M_1$. By the first part of the proof, $C$ is a $\Sigma_2$-even signature of $(M_2^*, \Gamma_2)$. Therefore $C \triangle D$ is a $\Sigma_2$-even cycle of $M_2$, hence a $\Sigma_1$-even cycle of $M_1$. Thus $D$ is a $\Sigma_1$-odd cycle of $M_1$.

We now proceed with the proof of (2). Let $D$ be a $\Sigma_1$-odd signature of $(M_1^*, \Gamma_1)$. Moreover, let $C$ be a $\Sigma_1$-odd cycle of $M_1$. Then $D \triangle C$ is a $\Sigma_1$-even signature of $(M_1^*, \Gamma_1)$. By part (1) and symmetry between $M_1$ and $M_2$, $D \triangle C$ is a $\Sigma_2$-odd cycle of $M_2$. Also, by part (1), $C$ is a $\Sigma_2$-even signature of $(M_2^*, \Gamma_2)$. Hence $D = (D \triangle C) \triangle C$ is a $\Sigma_2$-odd signature of $(M_2^*, \Gamma_2)$. Hence every $\Sigma_1$-odd signature of $(M_1^*, \Gamma_1)$ is a $\Sigma_2$-odd signature of $(M_2^*, \Gamma_2)$. The other inclusion follows by symmetry between $M_1$ and $M_2$.

### 3.1.4 Harmonious sets

A set of matroids $\{M_1, \ldots, M_n\}$ is harmonic if $M_i \neq M_j$, for all distinct $i, j \in [n]$, and there exist signatures $\Sigma_1, \ldots, \Sigma_n$ and $\Gamma_1, \ldots, \Gamma_n$ such that $\text{ecycle}(M_i, \Sigma_i) = \text{ecycle}(M_j, \Sigma_j)$ and $\text{ecycle}(M_i^*, \Gamma_i) = \text{ecycle}(M_j^*, \Gamma_j)$, for all $i, j \in [n]$.

**Theorem 15.** Suppose that $\{M_1, \ldots, M_n\}$ is a harmonious set of matroids. Then $n \leq 3$.

**Proof.** Suppose for a contradiction that there exists a harmonious set $\{M_1, \ldots, M_4\}$. Note that, by Proposition 11, $\Sigma_1, \ldots, \Sigma_4$, $\Gamma_1, \ldots, \Gamma_4$ are unique up to signature exchange. First suppose that $(M_k, \Sigma_k)$ is bipartite for some $k \in [4]$. Then, by Theorem 7, $(M_i^*, \Gamma_i)$ is bipartite for every $i \in [4] \setminus \{k\}$. Hence, for $i, j \in [4] \setminus \{k\}$, the matroids $M_i$ and $M_j$ have the same cocycles, hence $M_i = M_j$, a contradiction. Therefore, for every $i \in [4]$, $(M_i, \Sigma_i)$ is non-bipartite and by duality $(M_i^*, \Gamma_i)$ is non-bipartite as well. By Theorem 7, a cocycle $C$ of $M_4$ is non-preserved if and only
if it is $\Gamma_4$-odd. We fix $C$ to be an odd cocycle of $(M_4, \Gamma_4)$, and conclude that $C$ is non-preserved for $M_i$, for all $i \in [3]$. By definition, $C$ is an $M_4$-standard signature for $(M_i, \Sigma_i)$, for all $i \in [3]$.

For every $i \in [3]$, let $C_i$ be a $C$-odd signature of $(M_i^*, \Gamma_i)$. Note that such signatures exist because $(M_i, \Sigma_i)$ is non-bipartite, hence the symmetric difference of an odd circuit of $(M_i, \Sigma_i)$ and the signature of $(M_i^*, \Gamma_i)$ will be an odd signature. By Theorem 14(2), $C_i$ is a signature of $(M_i^*, \Gamma_4)$, for every $i \in [3]$. The symmetric difference of two signatures of $(M_i^*, \Gamma_4)$ is a cycle of $M_4$. Moreover, for some $j,k \in [3]$ with $j \neq k$, $C_j$ and $C_k$ have the same parity with respect to $\Sigma_4$. Hence $D := C_j \triangle C_k$ is a $\Sigma_4$-even cycle of $M_4$, so $D$ is a $\Sigma_i$-even cycle of $M_i$ for every $i \in [4]$. Therefore $C_j = D \triangle C_k$ is a $C$-odd signature of both $(M_j^*, \Gamma_j)$ and $(M_k^*, \Gamma_k)$. Now let $C'$ be a $\Sigma_4$-odd cycle of $M_4$. By Theorem 14(1), $C'$ is a $C$-even signature of $(M_j^*, \Gamma_j)$ and $(M_k^*, \Gamma_k)$. Therefore $C_j \triangle C'$ is a $C$-odd cycle of both $M_j$ and $M_k$. Hence, by Remark 12, $M_j = M_k$, a contradiction. □

3.2 Applications to signed graphs and grafts

In this section we show how the results for signed matroids apply to signed graphs and grafts.

Remark 16. Let $(G, T)$ be a graft, let $\Gamma$ be a $T$-join of $G$ and let $M = \text{cut}(G)$.

(1) A cut of $G$ is $T$-even if and only if it is $\Gamma$-even. In particular, $\text{ecut}(G, T) = \text{ecycle}(M, \Gamma)$.

(2) A set of edges is a $T$-join of $G$ if and only if it is a signature of $(M, \Gamma)$.

Proof of Theorem 4. We begin with the proof of (1). We omit the cases when $(G_1, \Sigma_1)$ or $(G_2, \Sigma_2)$ is bipartite. For $i = 1, 2$, let $M_i := \text{cycle}(G_i)$. By Theorem 7, there exist $\Gamma_1$ and $\Gamma_2$ such that $\text{ecycle}(M_1^*, \Gamma_1) = \text{ecycle}(M_2^*, \Gamma_2)$. Since $C_i$ is an odd cycle of $(M_i, \Sigma_i)$, it is non-preserved.
It follows from Remark 9(1) that \( C_i \) is a signature of \((M_{3-i}^*, \Gamma_{3-i})\). Hence, \( \text{ecycle}(M_1^*, C_2) = \text{ecycle}(M_2^*, C_1) \). Let \( T_i \) be the vertices of odd degree in \( G_i[C_{3-i}] \). Remark 16(1) implies that \( \text{ecut}(G_1, T_i) = \text{ecut}(G_2, T_i) \).

We proceed with the proof of (2). We omit the cases when \( T_1 = \emptyset \) or \( T_2 = \emptyset \). For \( i = 1, 2 \), let \( M_i := \text{cut}(G_i) \) and let \( \Gamma_i \) be a \( T_i \)-join of \( G_i \). Remark 16(1) implies that \( \text{ecycle}(M_1, \Gamma_1) = \text{ecycle}(M_2, \Gamma_2) \). By Theorem 7 there exist \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \) such that \( \text{ecycle}(M_1^*, \tilde{\Sigma}_1) = \text{ecycle}(M_2^*, \tilde{\Sigma}_2) \).

As \( \Sigma_i = \delta_{G_{3-i}}(t_{3-i}) \) is a \( T_{3-i} \)-odd cut of \( G_{3-i} \), by Remark 16(1) \( \Sigma_i \) is a \( \Gamma_{3-i} \)-odd cycle of \((M_{3-i}^*, \Gamma_{3-i})\). It follows from Remark 9(1) that \( \Sigma_i \) is a signature of \((M_i^*, \tilde{\Sigma}_i)\). We conclude that \( \text{ecycle}(G_1, \Sigma_i) = \text{ecycle}(M_1^*, \Sigma_1) = \text{ecycle}(M_2^*, \Sigma_2) = \text{ecycle}(G_2, \Sigma_2) \). 

Suppose that \( \text{ecycle}(G_1, \Sigma_i) = \text{ecycle}(G_2, \Sigma_2) \) and \( \text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2) \), where \( \text{cycle}(G_1) \neq \text{cycle}(G_2) \). If \((G_1, \Sigma_1)\) is bipartite, let \( \Sigma := \emptyset \). Otherwise, by Remark 13, there exists a \( T_2 \)-odd cut \( D \) of \((G_2, T_2)\); let \( \Sigma := D \). Then \( \Sigma \) is a standard signature of \((G_1, \Sigma_1)\). Given a signature \( \tilde{\Sigma}_i \) of \((G_i, \Sigma_i)\), \( \Sigma_i \Delta \tilde{\Sigma}_i \) is a cut \( D \) of \( G_i \). We say that \( \tilde{\Sigma}_i \) is \( T_i \)-even (resp. \( T_i \)-odd) if \( D \) is a \( T_i \)-even (resp. \( T_i \)-odd) cut.

**Proposition 17.** Suppose that \( \text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2) \) and \( \text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2) \), where \( \text{cycle}(G_1) \neq \text{cycle}(G_2) \). If \( \Sigma_1 \) and \( \Sigma_2 \) are standard signatures, the following hold.

1. Supposes that \((G_1, \Sigma_1)\) is non-bipartite. Then
   
   \( D \) is a \( \Sigma_1 \)-odd cycle of \( G_1 \) if and only if \( D \) is a \( \Sigma_2 \)-even \( T_2 \)-join of \( G_2 \).

2. Suppose that \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) are non-bipartite. Then
   
   \( D \) is a \( \Sigma_1 \)-odd \( T_1 \)-join of \( G_1 \) if and only if \( D \) is a \( \Sigma_2 \)-odd \( T_2 \)-join of \( G_2 \).

3. Suppose that \( T_1 \neq \emptyset \). Then
   
   \( D \) is a \( T_1 \)-odd cut of \( G_1 \) if and only if \( D \) is \( T_2 \)-even signature of \((G_2, \Sigma_2)\).
(4) Suppose that $T_1, T_2 \neq \emptyset$. Then

$D$ is a $T_1$-odd signature of $(G_1, \Sigma_1)$ if and only if $D$ is $T_2$-odd signature of $(G_2, \Sigma_2)$.

We illustrate Proposition 17 on the example in Figure 3. We have that $\Sigma'_1 := \delta_{G_3}(f) = \{1, 9\}$ is a standard signature of $(G_1, \Sigma_1)$ and $\Sigma'_3 := \delta_{G_1}(a) = \{1, 2\}$ is a standard signature of $(G_3, \Sigma_3)$. Then the odd cycle $\{4, 7, 9\}$ of $(G_1, \Sigma'_1)$ is a $\Sigma'_3$-even $T_3$-join of $G_3$. The set $\{1\}$ is a $\Sigma'_i$-odd $T_1$-join of $G_1$ and a $\Sigma'_3$-odd $T_3$-join of $G_3$. Moreover $\{1, 3, 5\} = \delta_{G_1}(\{a, c\})$ is a $T_1$-odd cut of $G_1$. As $\{1, 3, 5\} \triangle \Sigma'_3 = \{2, 3, 5\} = \delta_{G_3}(c)$, $\{1, 3, 5\}$ is a $T_3$-even signature of $(G_3, \Sigma'_3)$. Finally, $\{2, 9\}$ is a $T_1$-odd signature of $(G_1, \Sigma'_1)$ which is also a $T_3$-odd signature of $(G_3, \Sigma'_3)$.

**Proof of Proposition 17.** We prove parts (1) and (3) only, as statements (2) and (4) follow similarly from Theorem 14(2). We begin with the proof of (1). For $i = 1, 2$, let $M_i := \text{cycle}(G_i)$. Clearly, $D$ is a cycle of $G_1$ if and only if $D$ is a cycle of $M_1$. Let $\Gamma_2$ be a $T_2$-join of $G_2$. Remark 16(2) implies that $D$ is a $T_2$-join of $G_2$ if and only if $D$ is a signature of $(M^*_2, \Gamma_2)$. The result now follows from Theorem 14(1). We proceed with the proof of (3). For $i = 1, 2$, let $M_i := \text{cut}(G_i)$ and let $\Gamma_i$ be a $T_i$-join of $G_i$. Remark 16(1) implies that $D$ is a $T_1$-odd cut of $G_1$ if and only if $D$ is a $\Gamma_1$-odd cycle of $M_1$. Since $\Sigma_2$ is a standard signature of $(M^*_2, \Sigma_2)$, $\Sigma_2$ is a $\Gamma_1$-odd cycle of $M_1$. It follows from Theorem 14(1) that $\Sigma_2$ is $\Gamma_2$-even. $D$ is a $T_2$-even signature of $(G_2, \Sigma_2)$ if and only if $D$ is a signature of $(M^*_2, \Sigma_2)$ where $\Sigma_2 \triangle D$ is $T_2$-even. Equivalently, by Remark 16(1), $\Sigma_2 \triangle D$ is $\Gamma_2$-even. As $\Sigma_2$ is $\Gamma_2$-even, this occurs if and only if $D$ is $\Gamma_2$-even. The result now follows from Theorem 14(1).
3.3 Whitney-flips and Lovász-flips - a unified view

In this section we explain how Whitney-flips and Lovász-flips arise from a simple matroid construction. We will apply the same construction to even cut matroids to derive an operation on grafts that preserves even cuts. Given a matroid $M$ and disjoint subsets $I$ and $J$ of $E(M)$, we denote by $M/I \setminus J$ the matroid obtained from $M$ by contracting $I$ and deleting $J$. We require the following observation.

**Lemma 18.** Let $M$ be a matroid and let $a, b, c, d$ denote distinct elements of $M$. Suppose that \{a, b, c, d\} is both a cycle and a cocycle of $M$. Then $M/\{a, b\} \setminus \{c, d\} = M \setminus \{a, b\}/\{c, d\}$.

**Proof.** Let $M_1 := M/\{a, b\} \setminus \{c, d\}$ and let $M_2 := M \setminus \{a, b\}/\{c, d\}$. We want to show that the cycles of $M_1$ are exactly the cycles of $M_2$. By symmetry between $M_1$ and $M_2$ (and between $\{a, b\}$ and $\{c, d\}$), it suffices to show that every cycle of $M_1$ is a cycle of $M_2$. Let $C$ be any cycle of $M_1$. Then there exists a cycle $D$ of $M$ such that $C \subseteq D \subseteq C \cup \{a, b\}$. Since $\{a, b, c, d\}$ is a cocycle of $M$ and $M$ is binary, $|D \cap \{a, b, c, d\}|$ is even. Hence, either none of $a, b$ are in $D$ or both of $a, b$ are in $D$. In the former case, $D = C$ and $C$ is cycle of $M_2$ as required. In the latter case, $D = C \cup \{a, b\}$. Since $\{a, b, c, d\}$ is a cycle of $M$, $D \triangle \{a, b, c, d\} = C \cup \{c, d\}$ is a cycle of $M$. It follows that $C$ is cycle of $M_2$. 

Given a graph $G$ and disjoint subsets $I$ and $J$ of $E(G)$, we denote by $G/I \setminus J$ the graph obtained from $G$ by contracting the edges in $I$ and deleting the edges in $J$. We need to define minors for signed graphs and grafts. Let $(G, \Sigma)$ be a signed graph and let $e \in E(G)$. Then $(G, \Sigma) \setminus e$ is defined as $(G \setminus e, \Sigma \setminus \{e\})$. $(G, \Sigma)/e$ is equal to $(G \setminus e, \emptyset)$ if $e$ is an odd loop of $(G, \Sigma)$; to $(G \setminus e, \Sigma)$ if $e$ is an even loop of $(G, \Sigma)$; otherwise $(G, \Sigma)/e$ is equal to $(G/e, \Gamma)$, where $\Gamma$ is any signature of $(G, \Sigma)$ which does not contain $e$. Let $(G, T)$ be a graft and let $e \in E(G)$ with endpoints $s$ and
t. Then \((G,T)/e\) is defined as \((G/e,T')\), where \(T' := T \setminus \{s,t\}\) if both of \(s,t\) or none of \(s,t\) are in \(T\), and \(T' := T \setminus \{s,t\} \cup \{u\}\), where \(u\) is the vertex corresponding to the edge \(e\) in \(G/e\), if exactly one of \(s\) and \(t\) are in \(T\). \((G,T) \setminus e\) is equal to \((G \setminus e,\emptyset)\) if \(e\) is an odd cut of \((G,T)\); otherwise \((G,T) \setminus e\) is equal to \((G \setminus e,T)\). The following are easy to verify:

\[
\text{cycle}(G)/I \setminus J = \text{cycle}(G/I \setminus J)
\]

\[
\text{cycle}(G,\Sigma)/I \setminus J = \text{cycle}(G,\Sigma/I \setminus J)
\]

(1)

\[
\text{ecut}(G,T)/I \setminus J = \text{ecut}(G,T) \setminus I \setminus J
\]

3.3.1 Whitney-flips

Consider a graph \(G\) which consists of components \(G[X_1]\) and \(G[X_2]\), for some partition \(X_1,X_2\) of \(E(G)\). For \(i = 1,2\), pick vertices \(s_i\) and \(t_i\) in \(G[X_i]\). Denote by \(C\) the set of edges \(\{a,b,c,d\}\), where \(a = (s_1,t_1), b = (s_2,t_2), c = (s_1,t_2)\) and \(d = (s_2,t_1)\). Let \(H\) be the graph obtained from \(G\) by adding the edges in \(C\). Since \(C\) is a circuit and a cut of \(H\), it is a cycle and a cocycle of \(\text{cycle}(H)\). Lemma 18 implies that \(\text{cycle}(H) \setminus \{a,b\}/\{c,d\} = \text{cycle}(H)/\{a,b\} \setminus \{c,d\}\). It follows from (1) that \(\text{cycle}(H \setminus \{a,b\}/\{c,d\}) = \text{cycle}(H/\{a,b\} \setminus \{c,d\})\). It can now be easily verified that \(H \setminus \{a,b\}/\{c,d\}\) and \(H/\{a,b\} \setminus \{c,d\}\) are related by a Whitney-flip and that any two graphs related by a single Whitney-flip can be obtained in that way. In particular, graphs related by Whitney-flips have the same set of cycles.

3.3.2Lovász-flips

Consider a graph \(G\). Pick vertices \(s_1,t_1,s_2\) and \(t_2\) of \(G\). Denote by \(C\) the set of edges \(\{a,b,c,d\}\), where \(a = (s_1,t_1), b = (s_2,t_2), c = (s_1,t_2)\) and \(d = (s_2,t_1)\). Let \(H\) be the graph obtained from
by adding the edges in $C$. Since $C$ is an even cycle of $(H,C)$, it is a cycle of $\text{ecycle}(H,C)$. Since $C$ is a signature of $(H,C)$, it is a cocycle of $\text{ecycle}(H,C)$ (see Remark 8). Lemma 18 implies that $\text{ecycle}(H,C) \setminus \{a,b\}/\{c,d\} = \text{ecycle}(H,C)/\{a,b\} \setminus \{c,d\}$. It follows from (1) that $\text{ecycle}((H,C) \setminus \{a,b\}/\{c,d\}) = \text{ecycle}((H,C)/\{a,b\} \setminus \{c,d\})$. It can now be easily verified that $(H,C) \setminus \{a,b\}/\{c,d\}$ and $(H,C)/\{a,b\} \setminus \{c,d\}$ are related by a Lovász-flip (and possibly signature exchanges) and that any two signed graphs related by a single Lovász-flip can be obtained in that way. In particular, graphs related by Lovász-flips have the same set of even cycles.

3.3.3 A corresponding operation for even cuts

Let us now find a counterpart to the Lovász-flip operation for even cuts. Consider a graph $G$ which consists of components $G[X_1]$ and $G[X_2]$, for some partition $X_1,X_2$ of $E(G)$. For $i = 1,2$, pick vertices $s_i,t_i,u_i,v_i$ in $G[X_i]$. Denote by $C$ the set of edges $\{a,b,c,d\}$, where $a = (s_1,s_2), b = (t_1,t_2), c = (u_1,u_2)$ and $d = (v_1,v_2)$. Let $H$ be the graph obtained from $G$ by adding the edges in $C$. Let $T := \{s_1,s_2,t_1,t_2,u_1,u_2,v_1,v_2\}$. Since $C$ is an even cut of $(H,T)$, it is a cycle of $\text{ecut}(H,T)$. Moreover, $C$ is a $T$-join of $H$. It follows from Remark 8 and Remark 16(2) that $C$ is a cocycle of $\text{ecut}(H,T)$. Lemma 18 implies that $\text{ecut}(H,T) \setminus \{a,b\}/\{c,d\} = \text{ecut}(H,T)/\{a,b\} \setminus \{c,d\}$. It follows from (1) that $\text{ecut}((H,T) \setminus \{a,b\}/\{c,d\}) = \text{ecut}((H,T)/\{a,b\} \setminus \{c,d\})$. Hence, the two grafts $(H,T) \setminus \{a,b\}/\{c,d\}$ and $(H,T)/\{a,b\} \setminus \{c,d\}$ have the same even cuts. This defines a new operation that preserves even cuts. It turns out, however, that this operation is a special case of the Tilt operation introduced by Gerards [2] for even cycles.
References


