Packing Disjoint A-paths with Specified Ends

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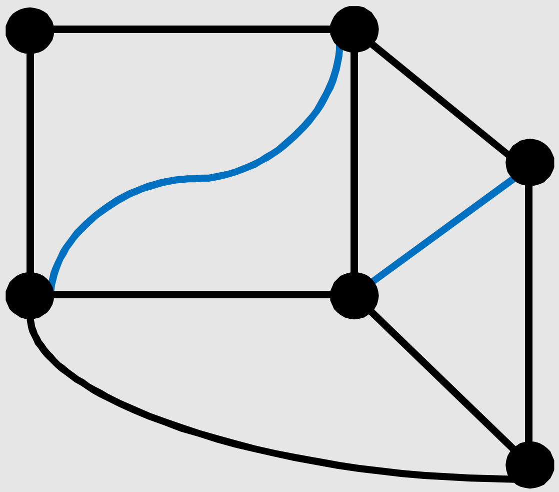
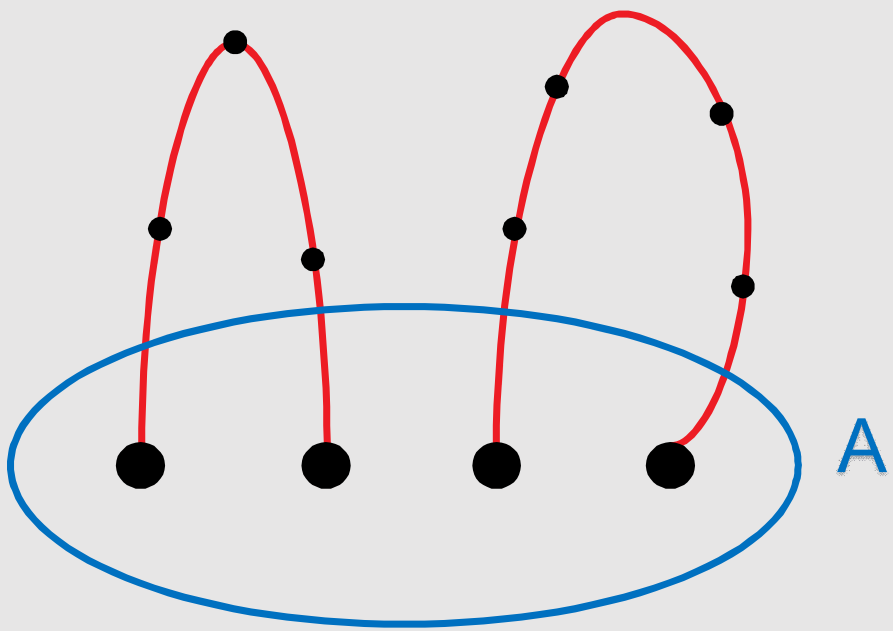
k-matching problem: does there exist
a matching of size k:



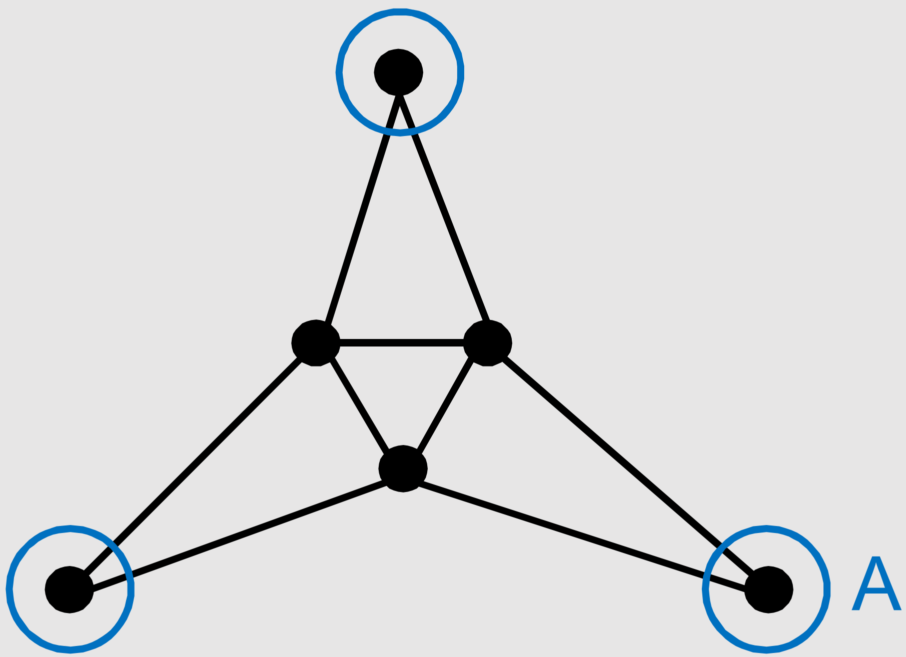
equivalent formulation: do there exist
k pairwise vertex disjoint paths?

Fix a subset A of V(G). An **A-path** is a path with both ends in A and no internal vertex in A.

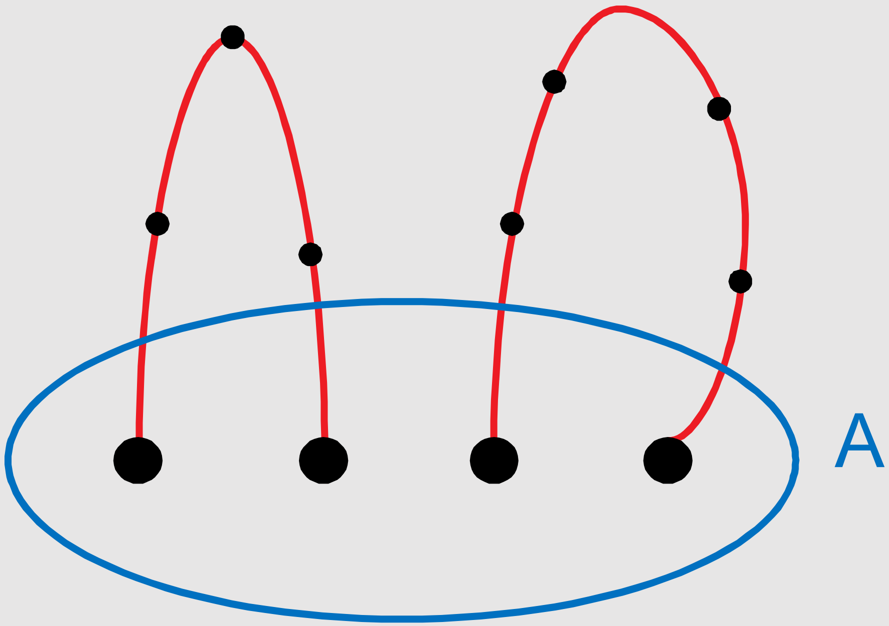


Question: do there exist k disjoint A-paths?

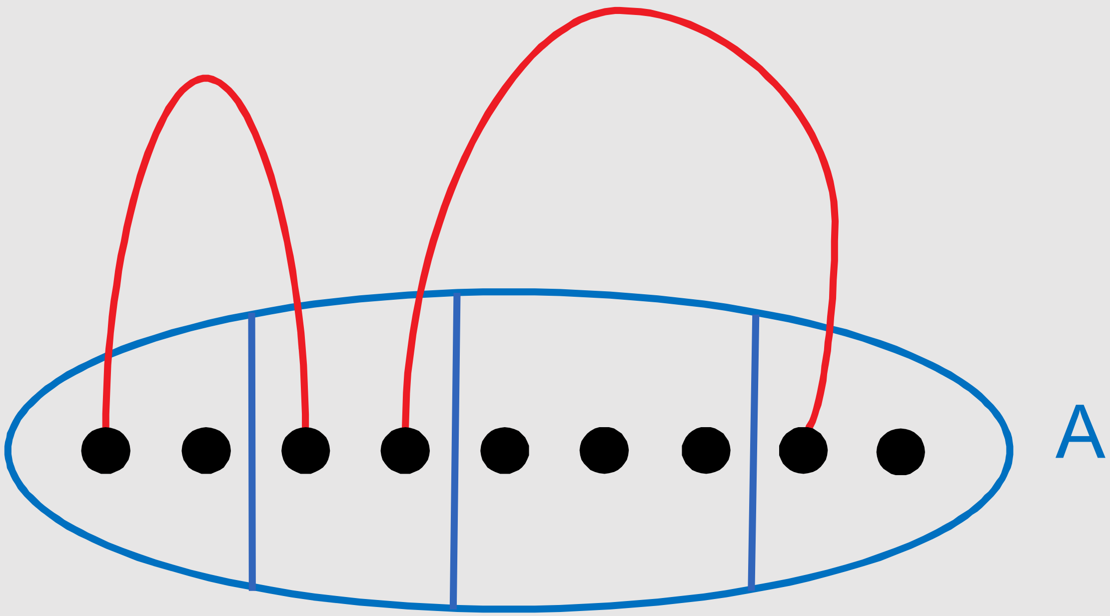
It is **not true** that the
max number of disjoint A-paths = min size hitting set



Theorem: (Gallai '62) there exists a min-max formula for the maximum number of disjoint A-paths.



A generalization: fix a partition of A and look for disjoint A-paths with ends in distinct subsets of the partition.



<u>Theorem</u>: (Mader '78) there exists a min-max formula for max number of disjoint A paths respecting a partition.

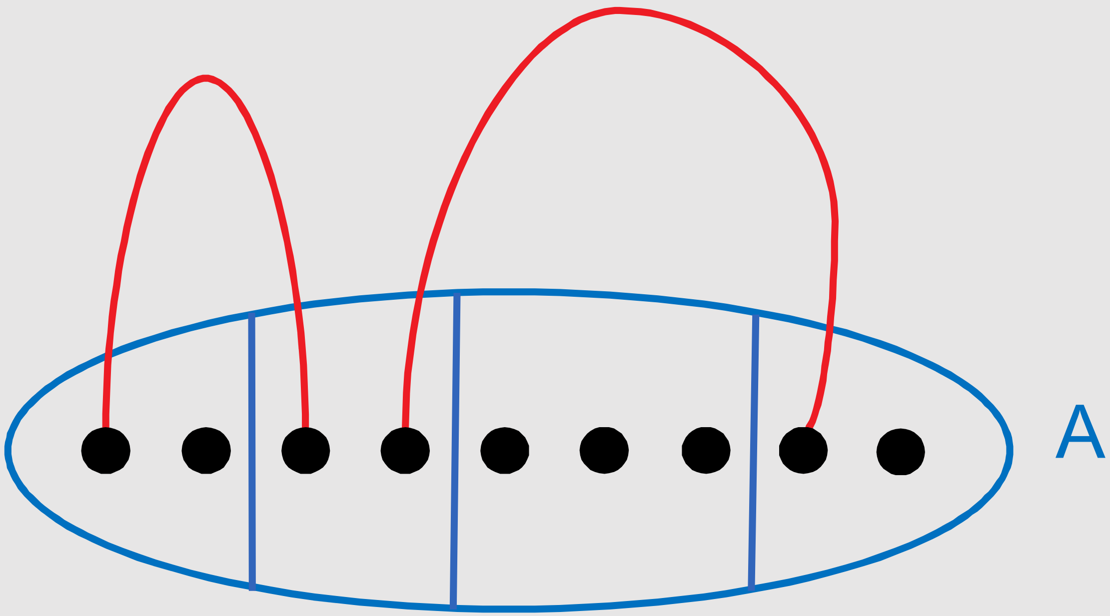
Other generalizations:

- Disjoint A-paths of odd length.

- (Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour ‘06) – Disjoint non-zero A-paths in group labeled graphs.

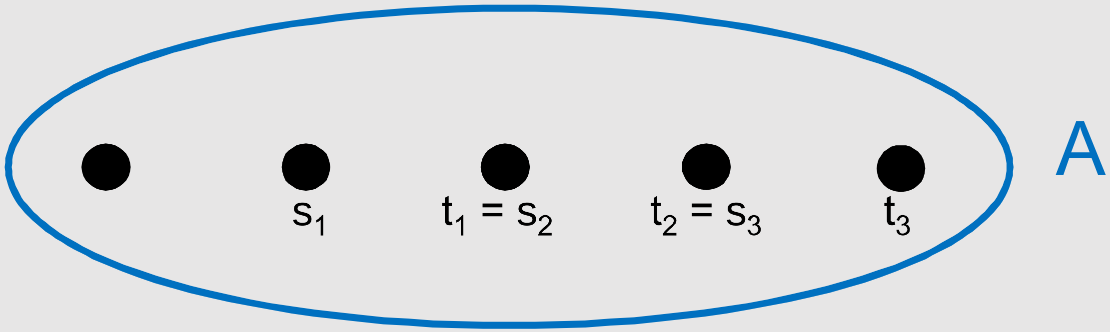
- (Wollan ’10) – Disjoint A-paths of non-zero length mod m.

A generalization: fix a partition of A and look for disjoint A-paths with ends in distinct subsets of the partition.

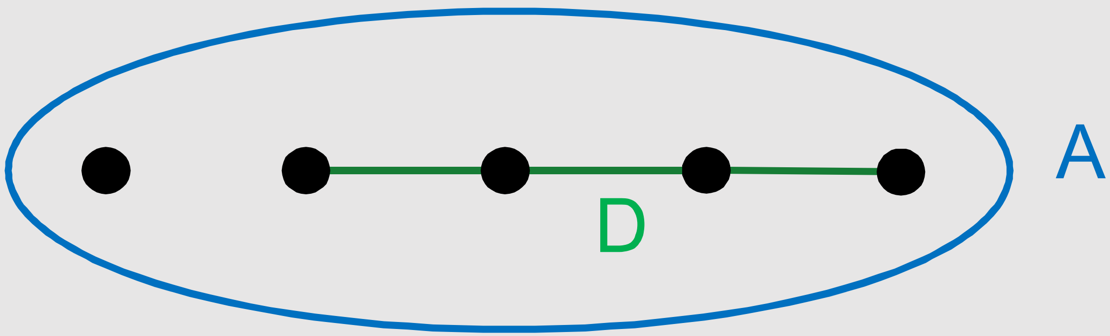


Theorem: (Mader '78) there exists a min-max formula for max number of disjoint A paths respecting a partition.

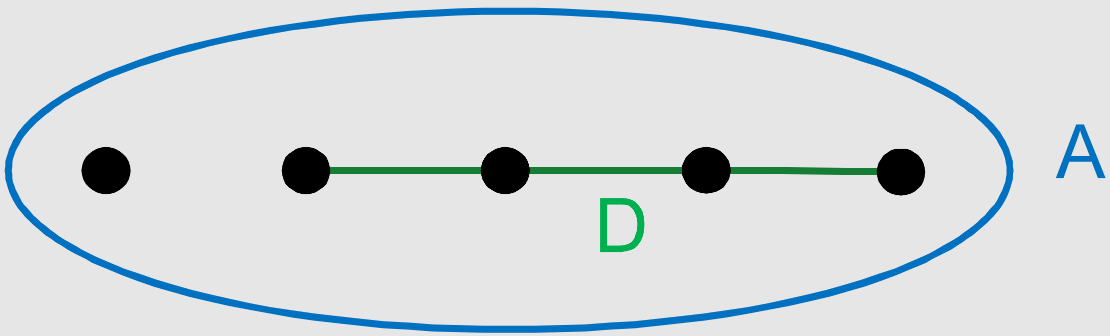
A new model: Given G, A a subset of V(G), and labels $\{(s_i, t_i): 1 \leq i \leq r\}$, do there exist k disjoint A-paths $P_1,\ldots,P_k$ such that the endpoints of Pi are s_j and t_j for some j



Think of the labels $\{(s_i, t_i): 1 \le i \le r\}$ as edges in an auxiliary graph D with vertex set A. Call it the **demand graph**.



Question: Given G, A a subset of V(G), and demand graph D with V(D) = A, do there exist k disjoint D-satisfying A-paths.



k-disjoint paths problem

Input: G and k pairs of vertices s_1, t_1,….,s_k, t_k.

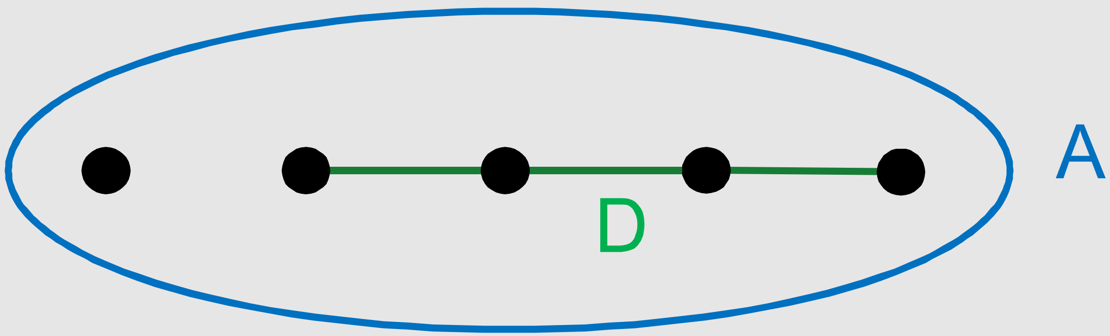
Output: k disjoint paths P_1,…,P_k such that P_i links s_i and t_i, or determine that no such paths exist.

Polynomial time algorithm for fixed k (RS 03), but very hard and complicated proof.

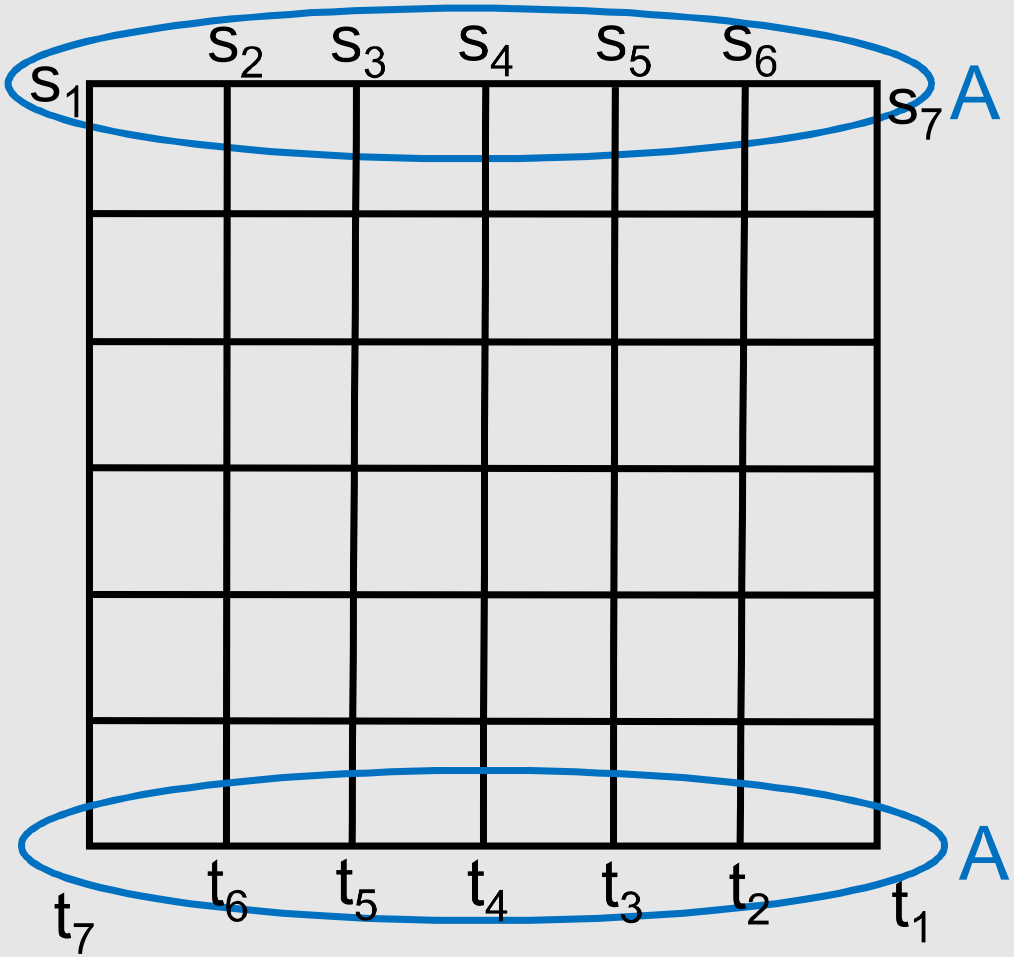
Conclusion: we can't expect a min-max formula.

Question 2: Does there exist a function f such that given G, a subset A of V(G) and D with V(D) = A, either

1. There exist k disjoint D-satisfying A-paths, or
2. There exist f(k) vertices hitting every D-satisfying A-path.



Question 2: **NO**



Let G be the r x r grid;
A is the first and last row;
D a matching of size r as shown.

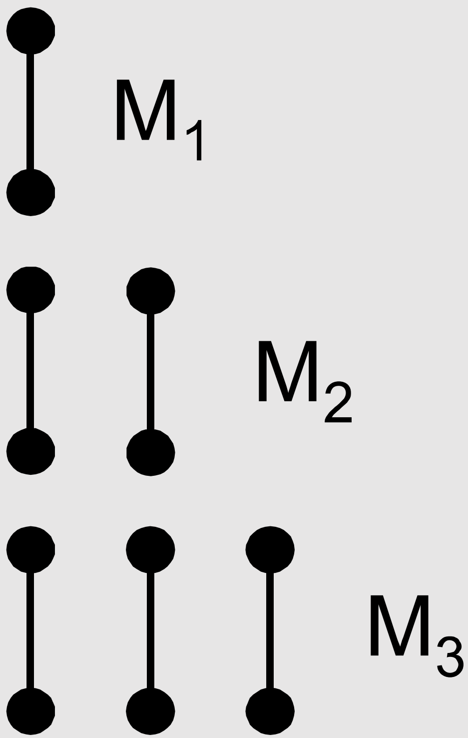
∄ 2 disjoint D-satisfying A-paths, but takes r/2 vertices to intersect them all.

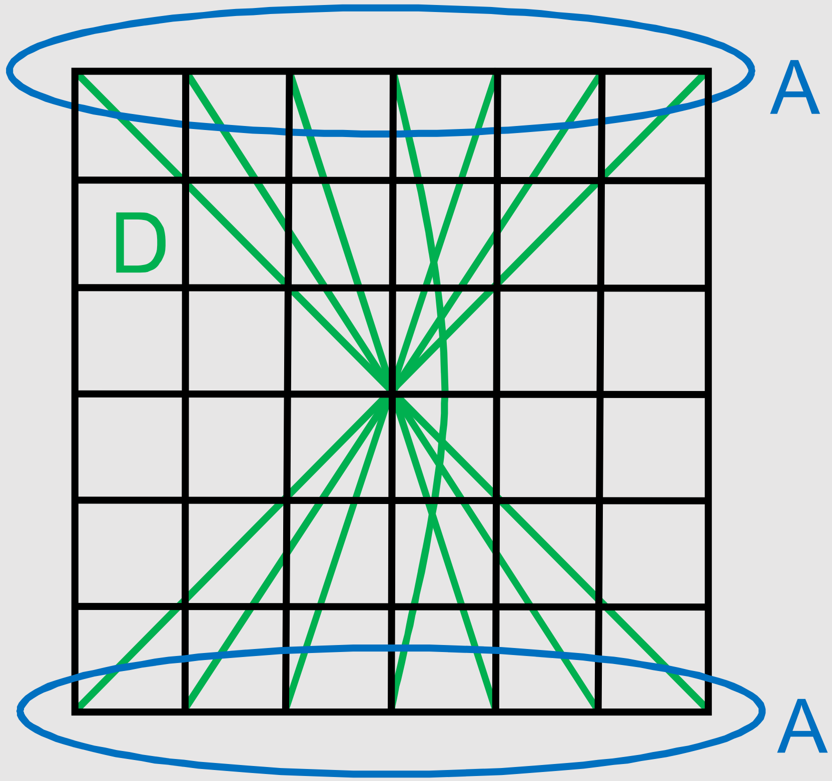
Question 3: Given a set **F** of graphs, does there exist a function f_F such that given G, a subset A in V(G) and D in **F** with V(D) = A, either

1. There exist k disjoint D-satisfying A-paths, or
2. There exist $f_F(k)$ vertices hitting every D-satisfying A-path.

Say F has the demand constrained A-path packing property - DCAPP

Prop: Let M_t be the matching with t edges. Then **M** = $\{M_t : t \geq 1\}$ does not have the demand-constrained A-path packing property.



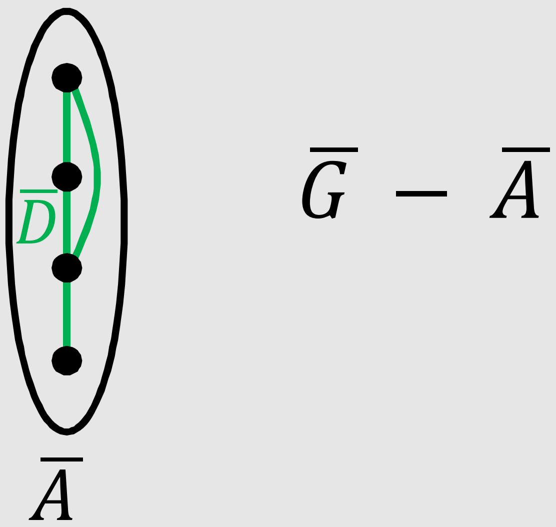


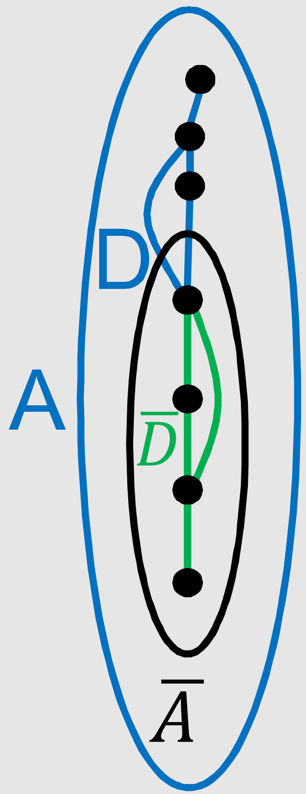
F a set of graphs. Let $\bar{\bar{F}}$ be the set of graphs obtained by taking all induced subgraphs of elements of **F**.

F has the demand constrained packing property if and only $\bar{\bar{F}}$ does.

Prop: **F** has the demand constrained packing property if and only $\bar{F}$ does.

Assume **F** has DCAPP. Let $\bar{D}$ be a graph in $\bar{F}$ with $\bar{D}$ a subgraph of D in **F**.





$\overline{G} - \overline{A}$

Add isolated vertices to A to get a new graph G with D a subgraph.

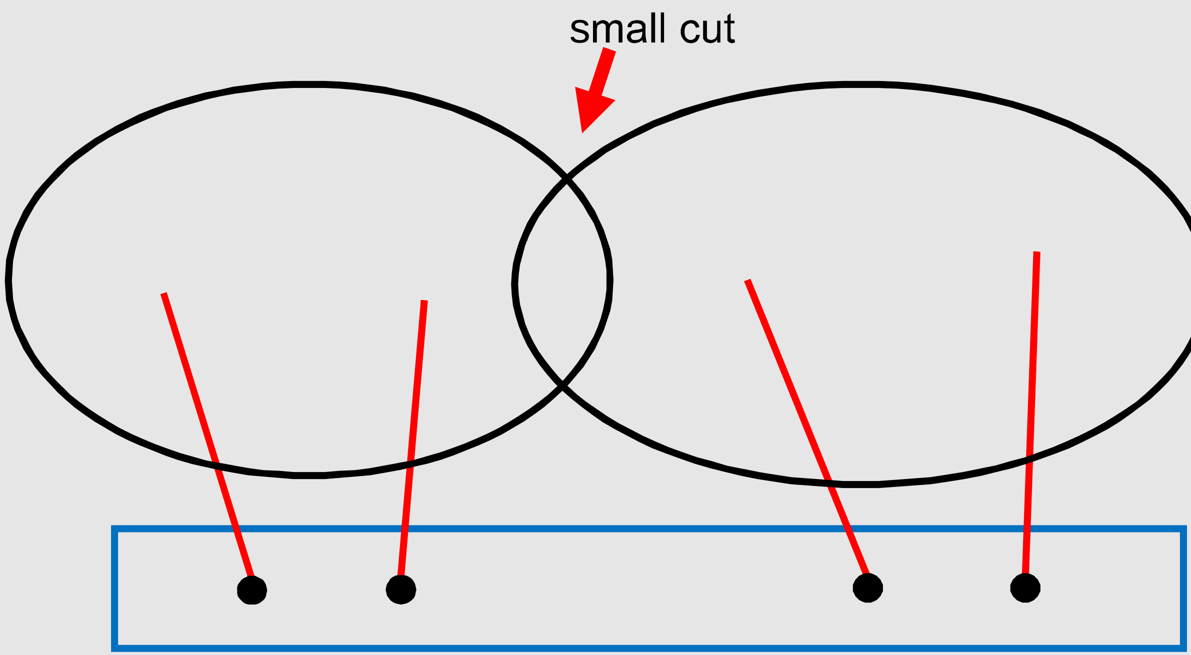
There exist k disjoint $\overline{D}$-satisfying paths in $\overline{G}$ if and only if there exist k disjoint D-satisfying paths in G.

Theorem (Marx, W) Let **F** be a set of graphs closed under taking induced subgraphs. **F** has the demand constrained A-path packing property if and only if there exists t such that M_t is not in **F**.

There is an algorithm which either finds the paths or hitting set running in time $g(k)n^c$ for some function g and constant c.

M_t the graph consisting of a matching with t edges.

Proof ideas: Pick a counterexample with k minimal.



There is no small cut in G - A separating two D-satisfying A-paths

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This defines a **tangle** in G-A.

The proof proceeds using concepts of tangles and tree-width but avoids many of the technicalities usually accompanying this type of argument.

Future directions: algorithmic questions

Can we characterize **F** such that given graph G, A, and D ∈ **F** with V(D) = A:

- In time $f_{\mathbf{F}}(k)n^c$, we can either find k D-satisfying A-paths or determine that they do not exist. (Theorem with Marx).
- In polynomial time (for arbitrary k) either find k D-satisfying A-paths or determine that they do not exist.