Packing Disjoint A-paths with Specified Ends

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k-matching problem: does there exist a matching of size $k$:

Equivalent formulation: do there exist $k$ pairwise vertex disjoint paths?
Fix a subset $A$ of $V(G)$. An **A-path** is a path with both ends in $A$ and no internal vertex in $A$.

Question: do there exist $k$ disjoint A-paths?
It is not true that the max number of disjoint A-paths = min size hitting set
Theorem: (Gallai ‘62) there exists a min-max formula for the maximum number of disjoint A-paths.
A generalization: fix a partition of $A$ and look for disjoint $A$-paths with ends in distinct subsets of the partition.

Theorem: (Mader ‘78) there exists a min-max formula for max number of disjoint $A$ paths respecting a partition.
Other generalizations:

- Disjoint A-paths of odd length.
- (Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour ‘06) – Disjoint non-zero A-paths in group labeled graphs.
- (Wollan ’10) – Disjoint A-paths of non-zero length mod m.
A generalization: fix a partition of $A$ and look for disjoint $A$-paths with ends in distinct subsets of the partition.

Theorem: (Mader ‘78) there exists a min-max formula for max number of disjoint $A$ paths respecting a partition.
A new model: Given G, A a subset of V(G), and labels \{\(s_i, t_i\): \(1 \leq i \leq r\}\), do there exist \(k\) disjoint A-paths \(P_1, \ldots, P_k\) such that the endpoints of \(P_i\) are \(s_j\) and \(t_j\) for some \(j\).
Think of the labels \( \{(s_i, t_i): 1 \leq i \leq r\} \) as edges in an auxiliary graph \( D \) with vertex set \( A \). Call it the demand graph.
Question: Given $G$, $A$ a subset of $V(G)$, and demand graph $D$ with $V(D) = A$, do there exist $k$ disjoint $D$-satisfying $A$-paths.
k-disjoint paths problem

**Input:** G and k pairs of vertices $s_1, t_1, \ldots, s_k, t_k$.

**Output:** k disjoint paths $P_1, \ldots, P_k$ such that $P_i$ links $s_i$ and $t_i$, or determine that no such paths exist.

Polynomial time algorithm for fixed k (RS 03), but very hard and complicated proof.

**Conclusion:** we can’t expect a min-max formula.
Question 2: Does there exist a function $f$ such that given $G$, a subset $A$ of $V(G)$ and $D$ with $V(D) = A$, either
1. There exist $k$ disjoint $D$-satisfying $A$-paths, or
2. There exist $f(k)$ vertices hitting every $D$-satisfying $A$-path.
Question 2: **NO**

Let $G$ be the $r \times r$ grid; $A$ is the first and last row; $D$ a matching of size $r$ as shown.

There are no 2 disjoint $D$-satisfying $A$-paths, but takes $r/2$ vertices to intersect them all.
Question 3: Given a set $\mathbf{F}$ of graphs, does there exist a function $f_{\mathbf{F}}$ such that given $G$, a subset $A$ in $V(G)$ and $D$ in $\mathbf{F}$ with $V(D) = A$, either

1. There exist $k$ disjoint $D$-satisfying $A$-paths, or

2. There exist $f_{\mathbf{F}}(k)$ vertices hitting every $D$-satisfying $A$-path.

Say $\mathbf{F}$ has the demand constrained $A$-path packing property - DCAPP
Prop: Let $M_t$ be the matching with $t$ edges. Then $\mathbf{M} = \{M_t : t \geq 1\}$ does not have the demand-constrained A-path packing property.
$\mathbf{F}$ a set of graphs. Let $\overline{F}$ be the set of graphs obtained by taking all induced subgraphs of elements of $\mathbf{F}$.

$\mathbf{F}$ has the demand constrained packing property if and only $\overline{F}$ does.
Prop: \( F \) has the demand constrained packing property if and only if \( \overline{F} \) does.

Assume \( F \) has DCAPP. Let \( \overline{D} \) be a graph in \( \overline{F} \) with \( \overline{D} \) a subgraph of \( D \) in \( F \).

\[
\overline{G} - \overline{A}
\]
Add isolated vertices to A to get a new graph $G$ with $D$ a subgraph.

There exist $k$ disjoint $\overline{D}$-satisfying paths in $\overline{G}$ if and only if there exist $k$ disjoint $D$-satisfying paths in $G$. 
**Theorem (Marx, W)** Let $F$ be a set of graphs closed under taking induced subgraphs. $F$ has the demand constrained A-path packing property if and only if there exists $t$ such that $M_t$ is not in $F$.

There is an algorithm which either finds the paths or hitting set running in time $g(k)n^c$ for some function $g$ and constant $c$.

$M_t$ the graph consisting of a matching with $t$ edges.
Proof ideas: Pick a counterexample with $k$ minimal.

There is no small cut in $G - A$ separating two $D$-satisfying $A$-paths
There is no small cut in G - A separating two D-satisfying A-paths.

This defines a **tangle** in G-A.

The proof proceeds using concepts of tangles and tree-width but avoids many of the technicalities usually accompanying this type of argument.
Future directions: algorithmic questions

Can we characterize $F$ such that given graph $G$, $A$, and $D \in F$ with $V(D) = A$:

- In time $f_F(k)n^c$, we can either find $k$ $D$-satisfying $A$-paths or determine that they do not exist. (Theorem with Marx).

- In polynomial time (for arbitrary $k$) either find $k$ $D$-satisfying $A$-paths or determine that they do not exist.