# A New Proof for the Weak-Structure Theorem with Explicit Constants

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**Question:** What do graphs with no $K_t$ minor look like?

Theorem: (RS '03) For every t, there exists an $\alpha = \alpha(t)$ such that every graph with no $K_t$ minor can be constructed by repeated $\alpha$ sums of graphs which are $\alpha$-near embedded in a surface $\Sigma$ in which $K_t$ does not embed

- Global tree-like structure with pieces based on a parameterized near-embeddings in a surface.

## Some difficulties to working with the structure theorem:

- Many technicalities and a long and difficult proof.
- Astronomical (and unknown) constants a drawback to algorithmic applications.

What if we want a local “well-behaved” subgraph instead of a global decomposition of the whole graph?

**Question:** What do graphs with:

no $K_t$ minor

look like?

**Answer 1:** they could have bounded tree-width.

A tree decomposition of a graph G breaks G up into a tree-structure of constant sized subsets of vertices.

Graphs of bounded tree-width are **nice**:

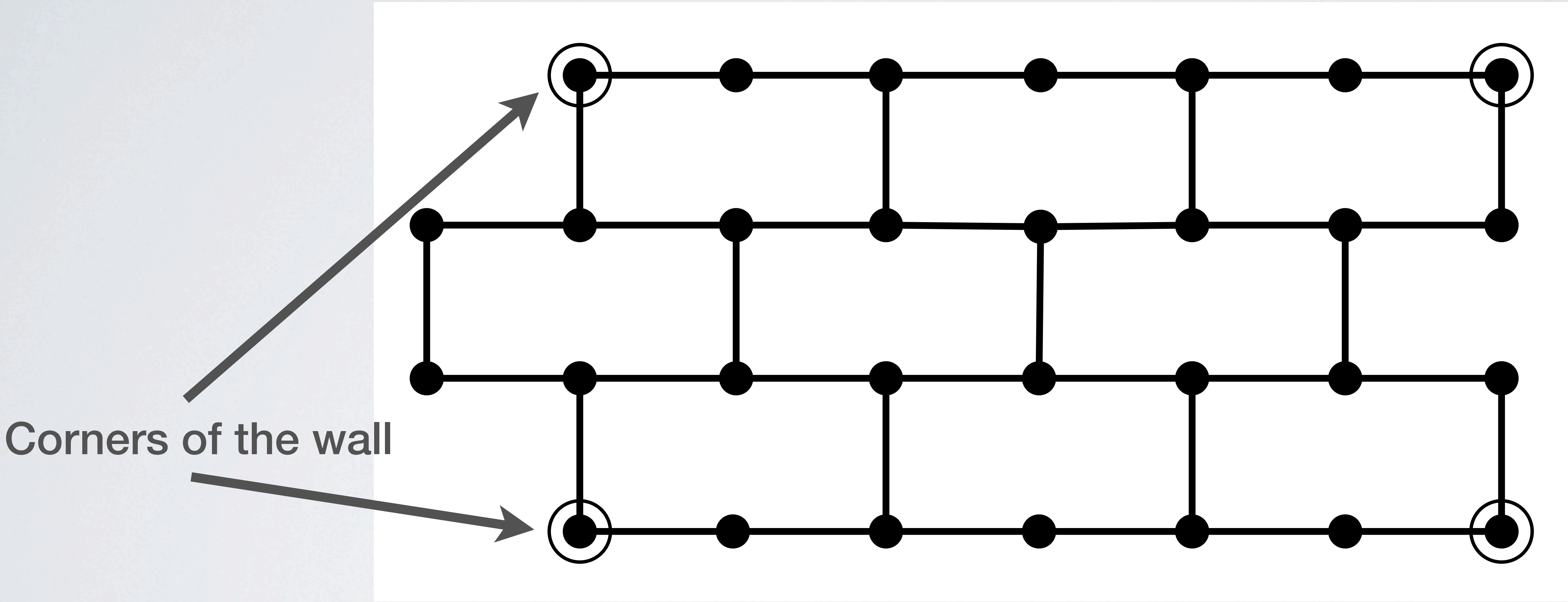
- simple decomposition;
- algorithmically well behaved.

Question: What do graphs with no $K_t$ minor and large tree-width look like?

This is the question answered by the Weak Structure Theorem of RS

- Basis of several important applications: RS minor testing algorithm, testing for subdivisions.
- New direct proof with explicit constants.

Theorem: (RS '86) For every r, there exists an **w** such that if a graph G has tree-width at least **w**, then G contains the $r \times r$ grid as a minor.

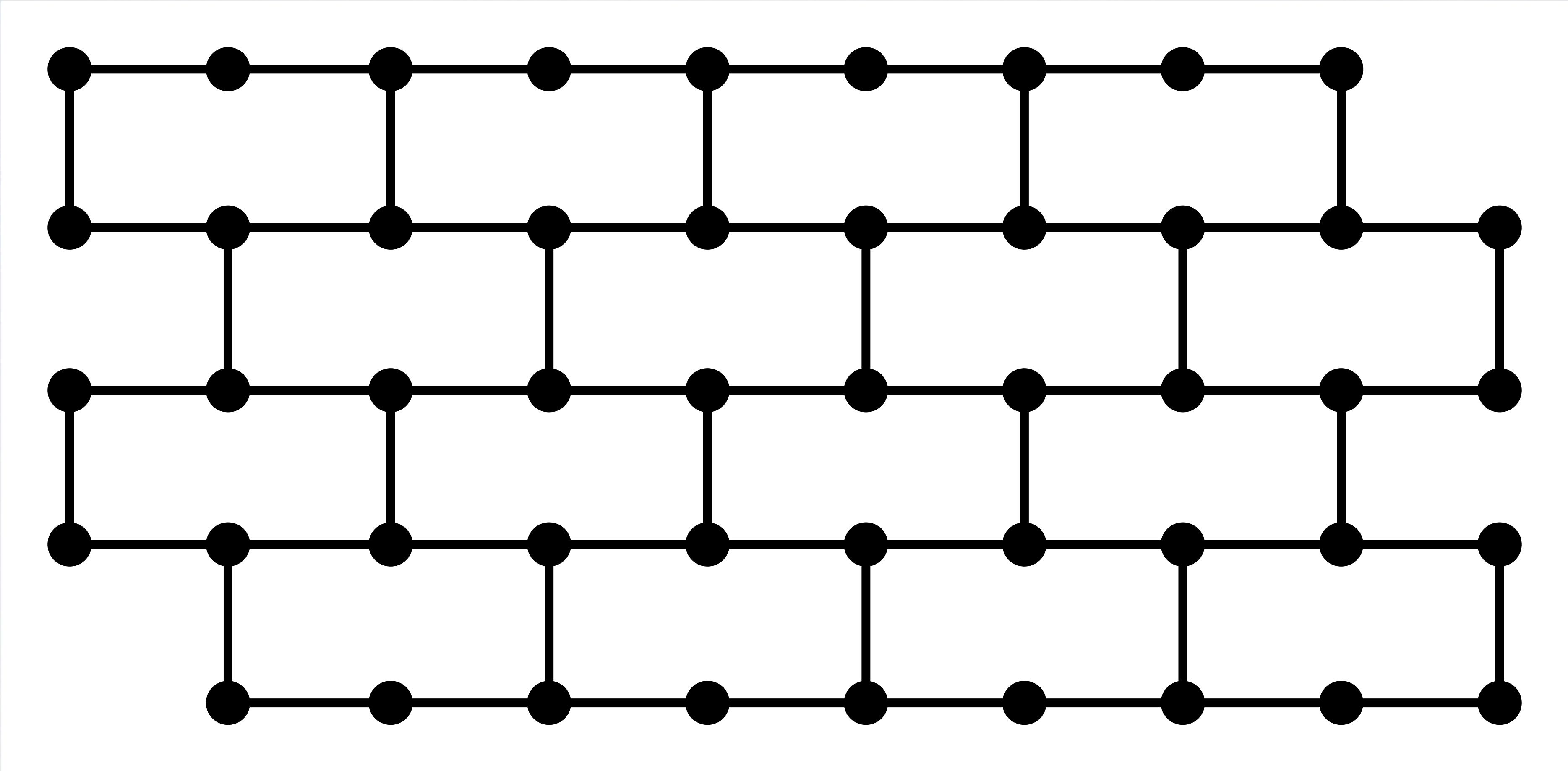



**r-wall** is obtained from $2r \times r$ grid by deleting odd vertical edges in the odd rows and even vertical edges in the even rows.

Corollary: For every r, there exists a **w** such that if a graph has tree-width at least **w**, then it contains as a subgraph a subdivision of an r-wall.

Assume G has no $K_t$ minor and does have a big r-wall subdivision subgraph W. How does G-W attach to W?

G has no $K_t$ minor and a subdivision of r-wall ($r >> t$).



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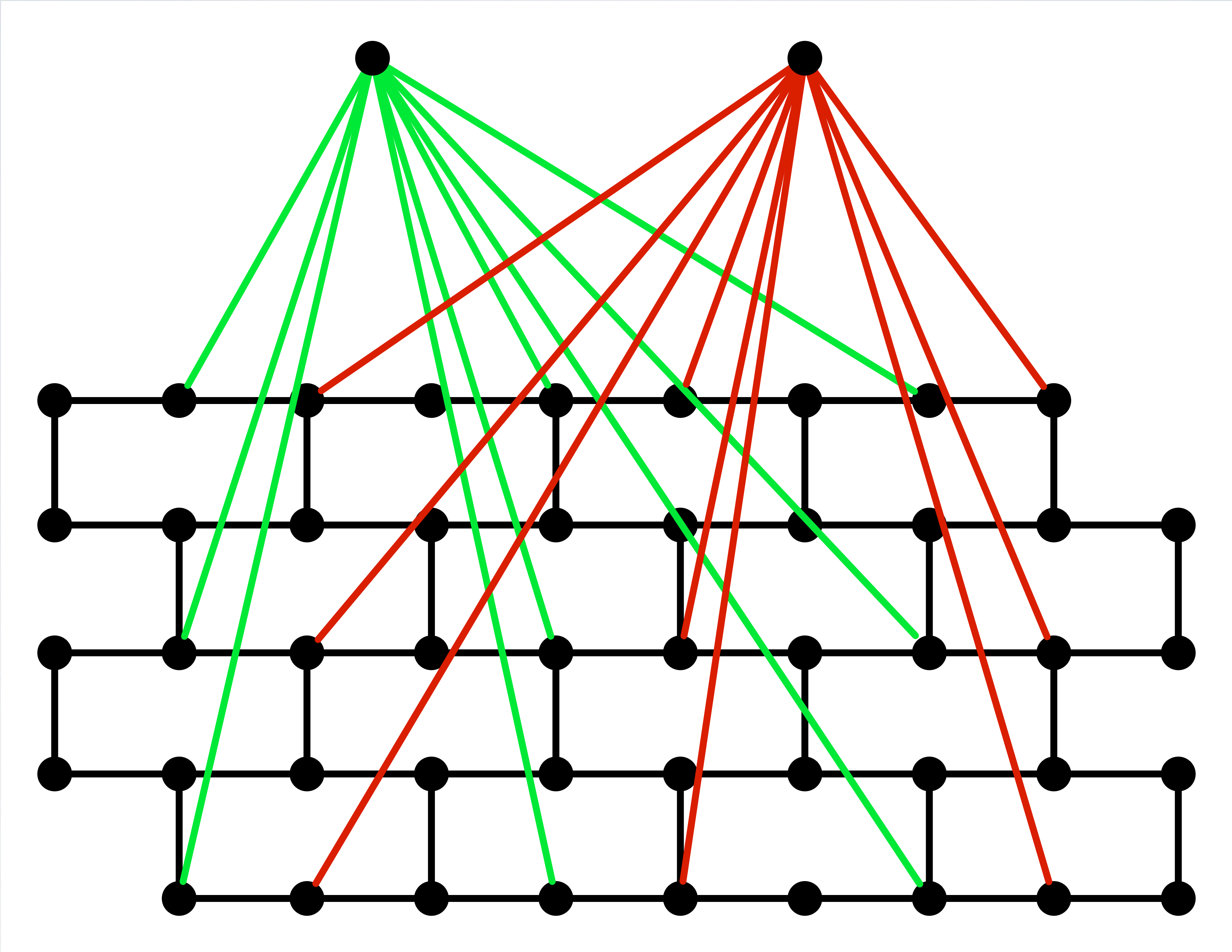
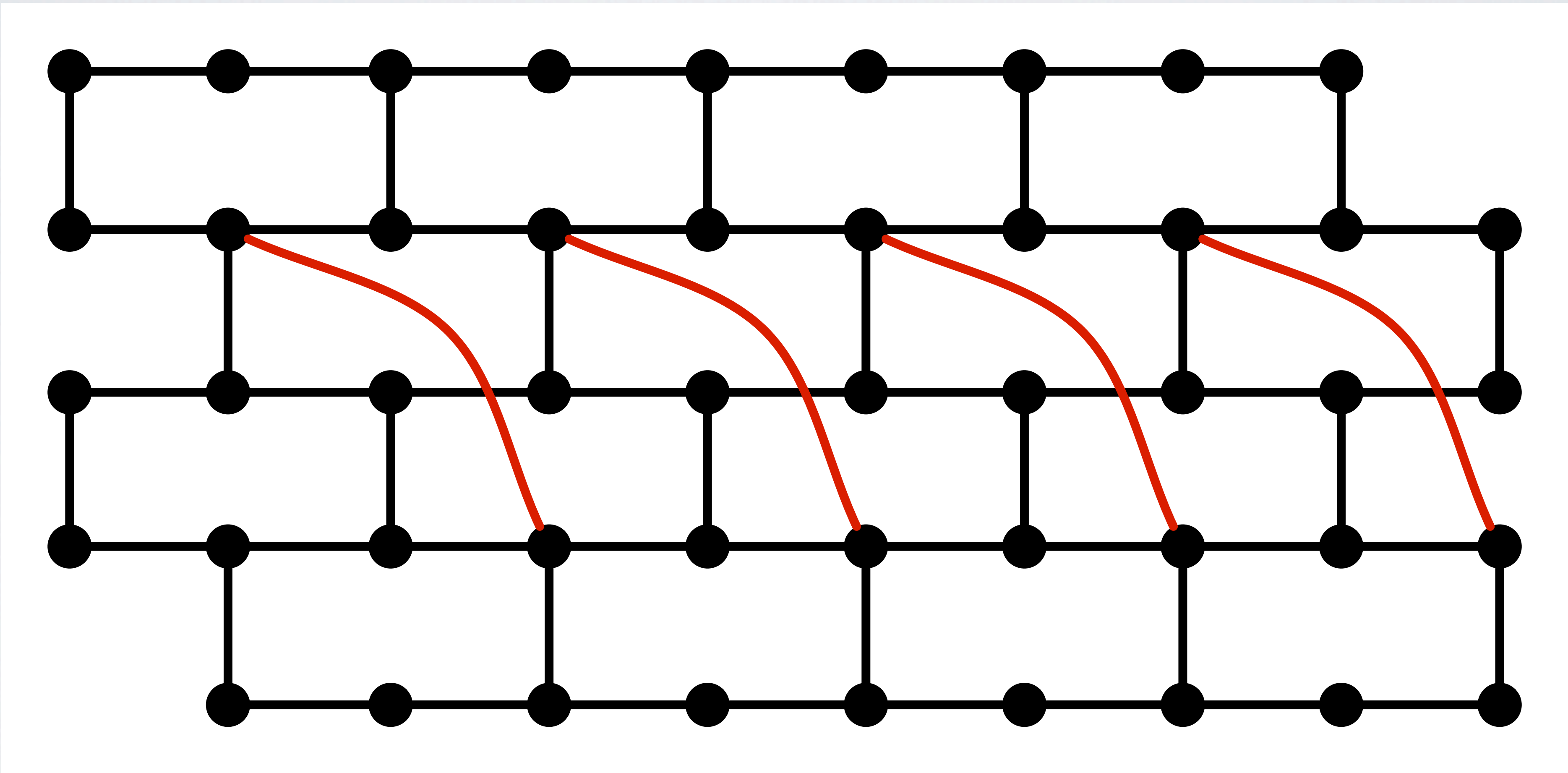


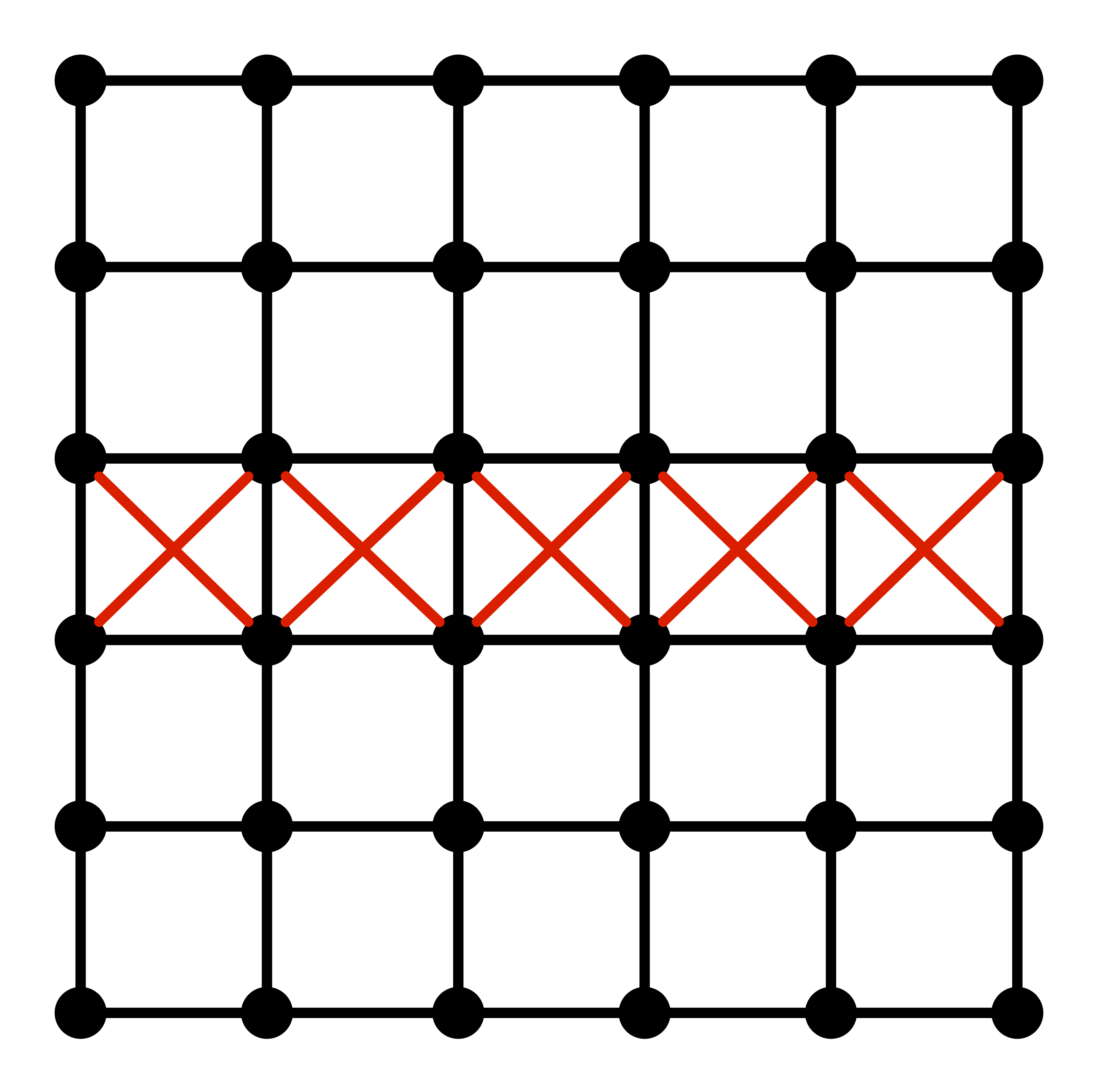
1. Many components of G-W have attachments all over W.

G has no $K_t$ minor and a subdivision of r-wall (r >>t).



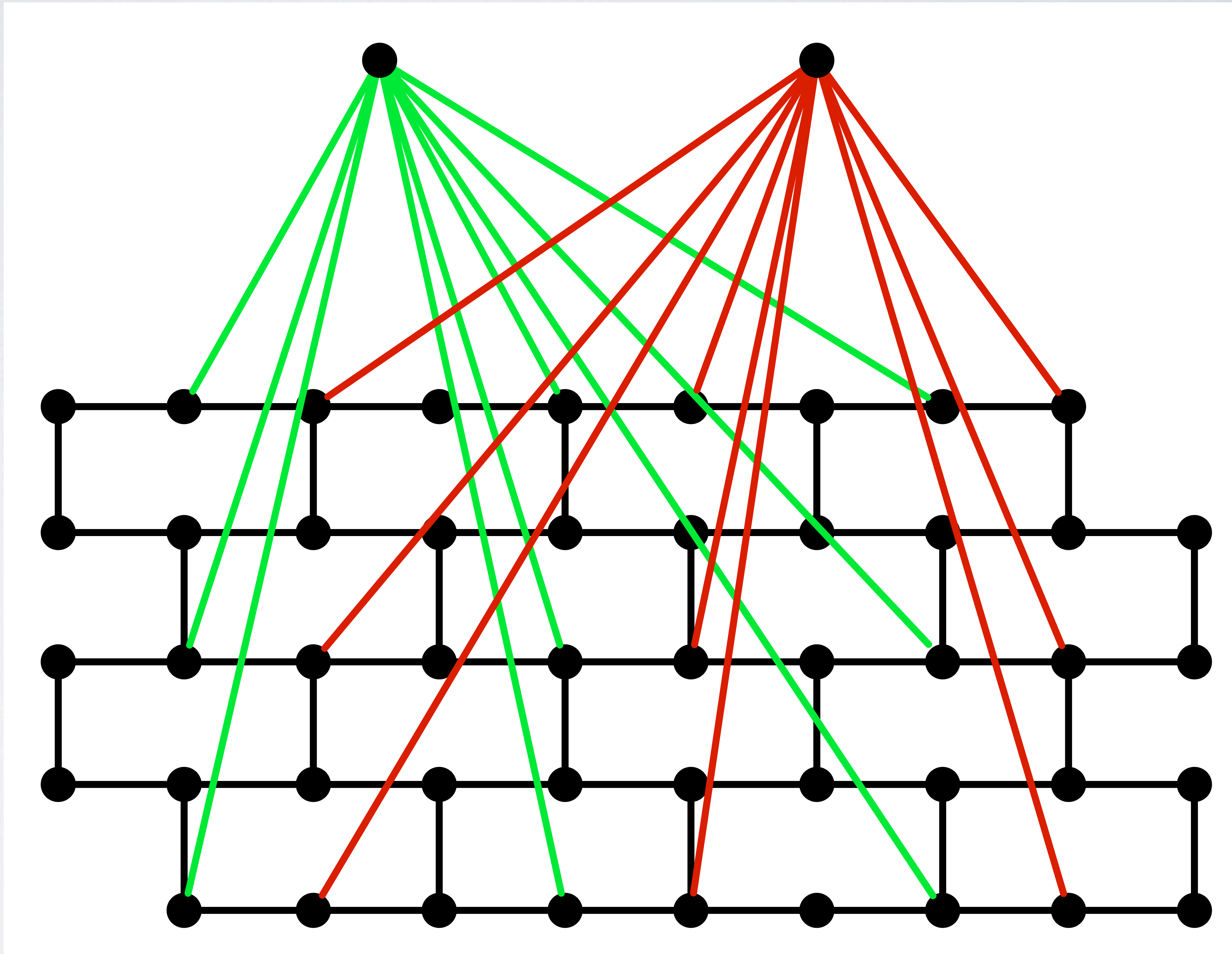
2. Many disjoint W-paths, each with endpoints in distinct faces.

If we have many such W-paths, $\exists$ an r’ × r’ grid minor with crosses in the middle row of faces.



$r' \geq t^2 \Rightarrow$ G contains $K_t$ as a minor, contradiction

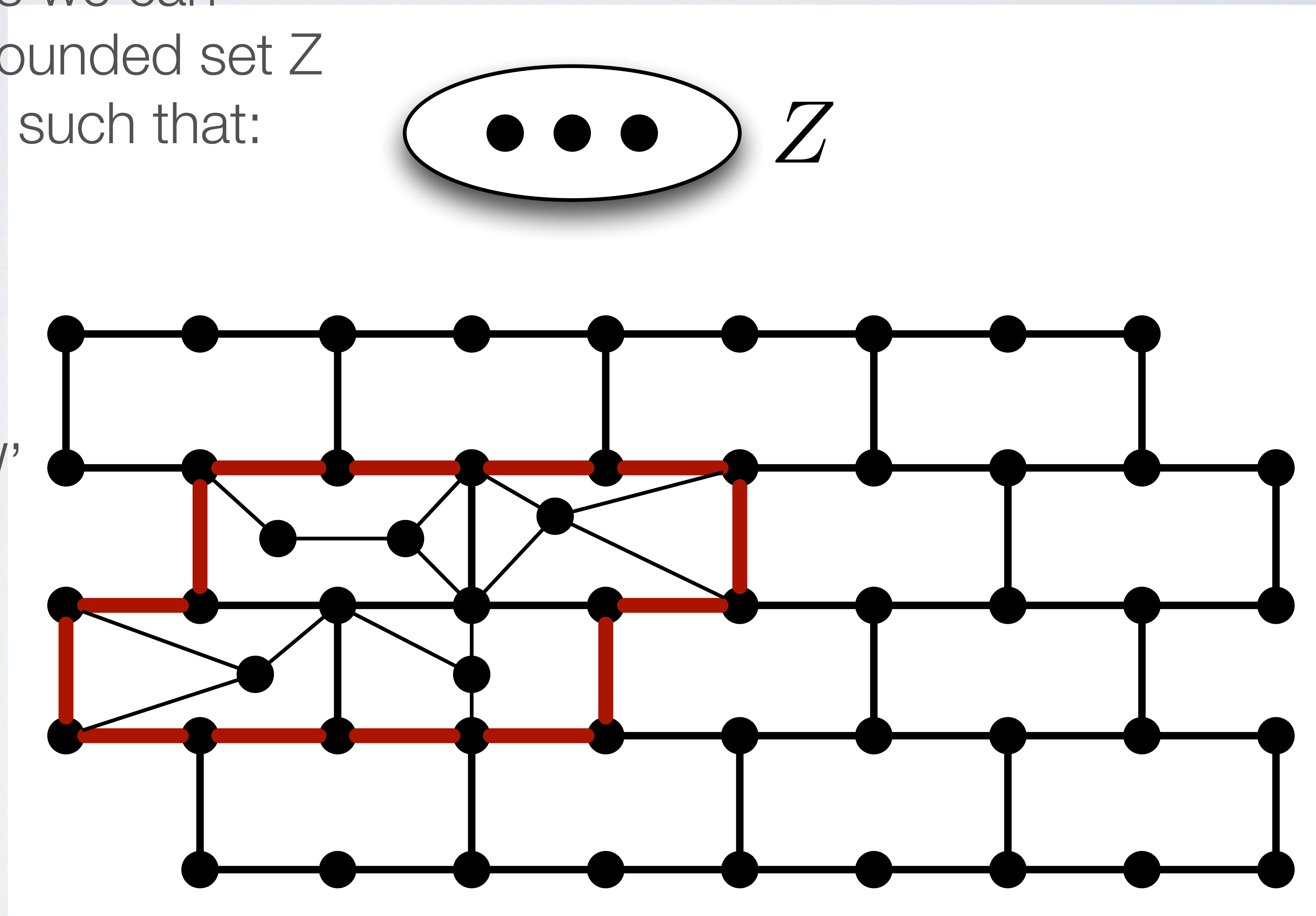
**Conclusion**: G-W must attach to W in a way that respects the planarity of W.



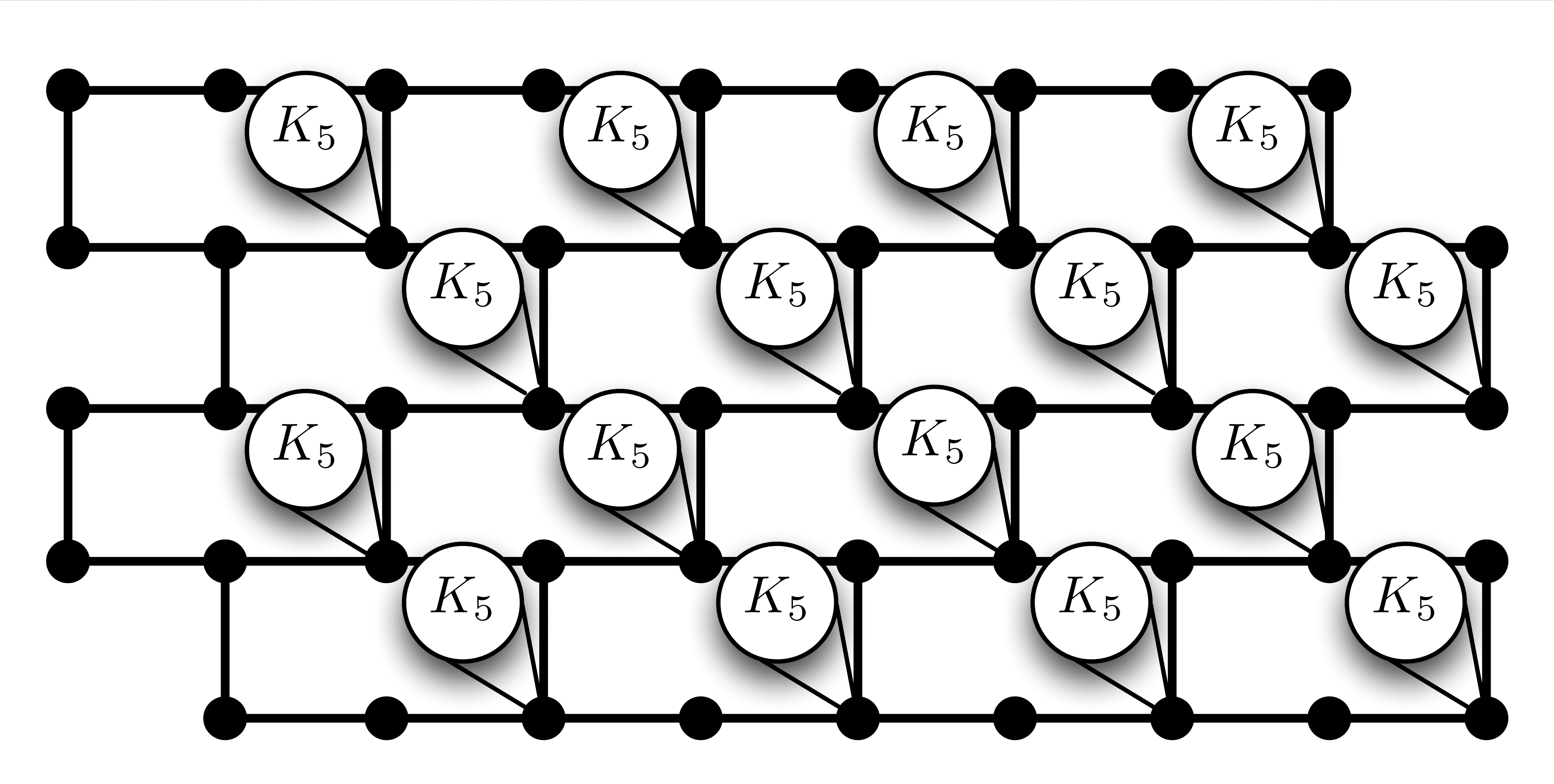
**Conclusion**: G-W must attach to W in a way that respects the planarity of W.

- Perhaps we can delete a bounded set Z of vertices such that:

- Find a subwall W' whose boundary separates "internal" planar subgraph in G-Z?

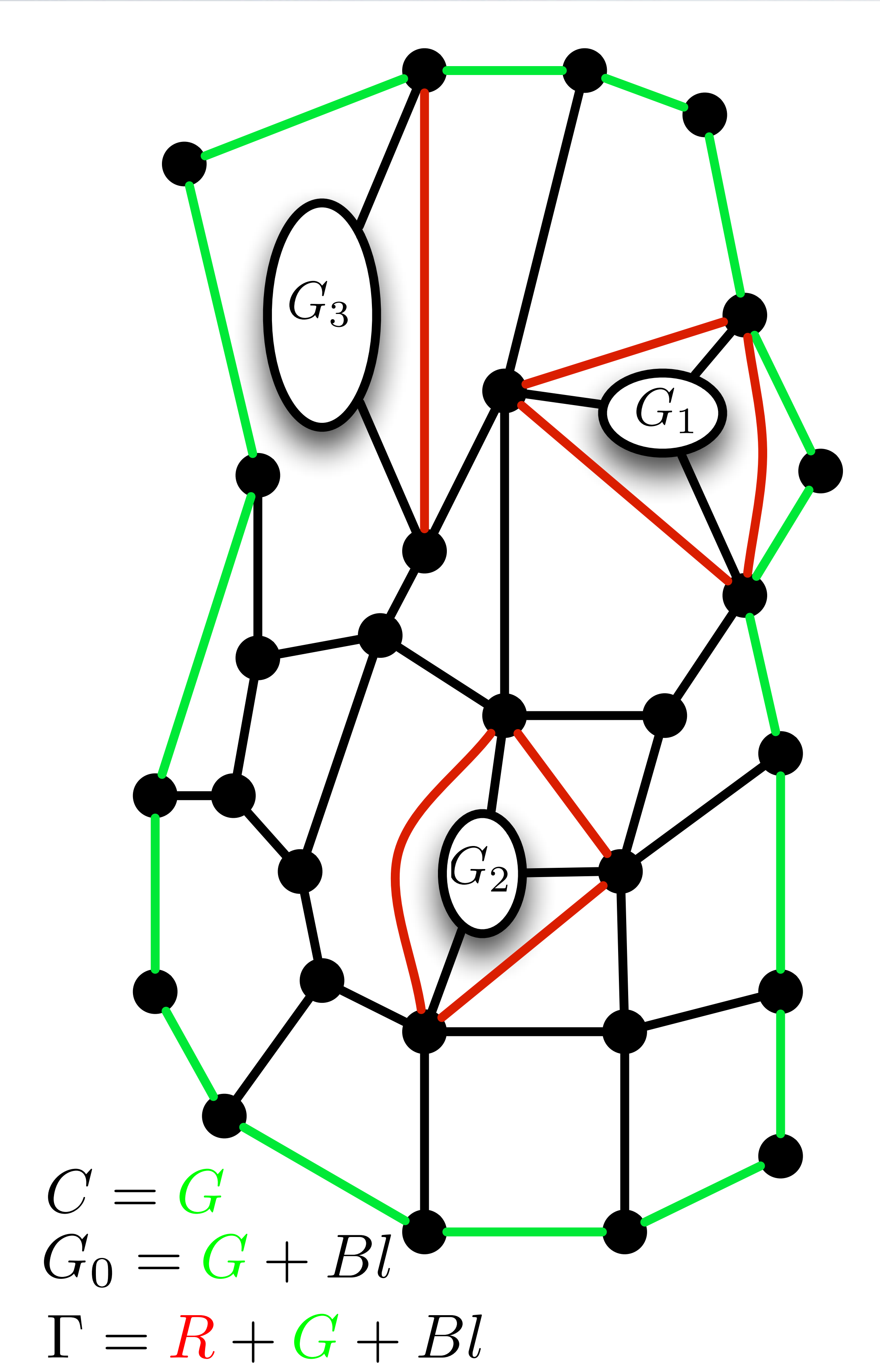


**No**: Take a wall and glue a $K_5$ clique onto each vertex.




- Can’t delete bounded number of vertices and find a genuinely planar subwall.

## **Conclusion**: We need “planarity up to ≤ 3 separations”.




C a cycle in G: G is **C-flat** if ∃ $G_0,\ldots,G_k$, Γ such that:

1. $G = G_0 \cup G_1 \cup \ldots\ldots \cup G_k$
2. $C \subseteq G_0$, Γ is a plane graph with $G_0 \subseteq \Gamma$ and $V(G_0) = V(\Gamma)$;
3. C bounds a face of Γ;
4. $|V(G_i) \cap V(G_0)| \leq 3$ and vertices of $V(G_i) \cap V(G_0)$ are pairwise adjacent and co-facial in Γ;
5. $G_i \cap G_j \subseteq V(G_0)$.

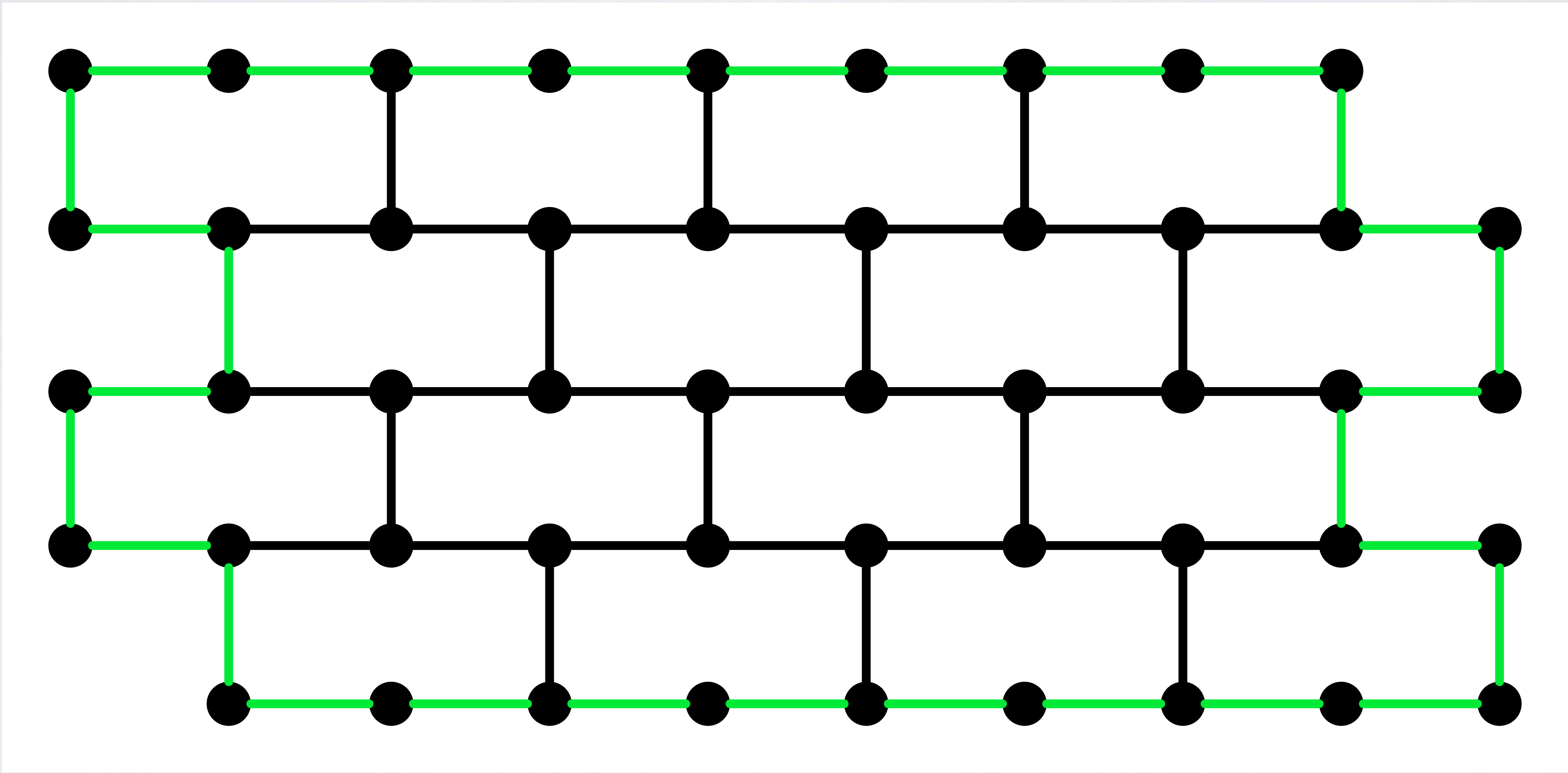
A wall $W$ with boundary cycle $C$ is **flat** in $G$ if there exists a separation $(A, B)$ such that

1. $A \cap B \subseteq V(C)$ and $W \subseteq G[B]$;

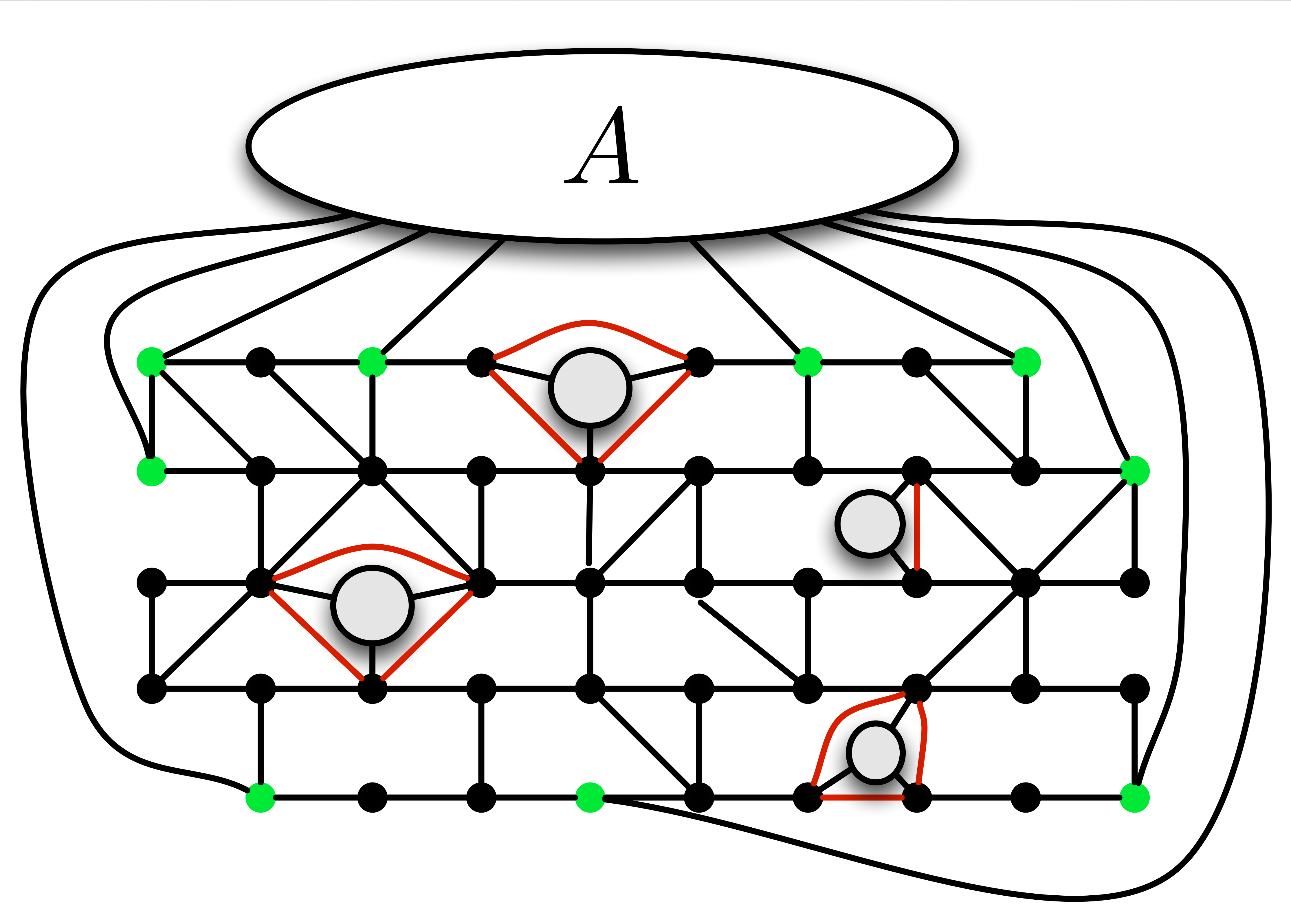
2. $C'$ is the cycle on $A \cap B$ given by $C$, then $G[B]$ is $C'$-flat.

It is non-trivially flat if the corners of $W$ are contained in $A \cap B$

# Example of a flat wall



# Example of a flat wall



Theorem: For every $r, t \geq 1$, $r$ even, let $R = 49152t^{24}(12t^2 + r)$, let $G$ be a graph and $W$ an $R$-wall in $G$. Then either $G$ has a $K_t$ minor (grasped by $W$) or there exists a set $X \subseteq V(G)$ with $|X| \leq 12288t^{24}$ and an $r$-subwall $W'$ of $W$ such that $V(W') \cap X = \emptyset$ and $W'$ is a non-trivial flat wall in $G-X$.

Robertson Seymour showed a qualitative version in GM 13 (Theorem 9.8).

Giannopoulou and Thilikos showed a version with a linear dependence in r for fixed t.

## Why “Weak Structure Theorem” - Applications

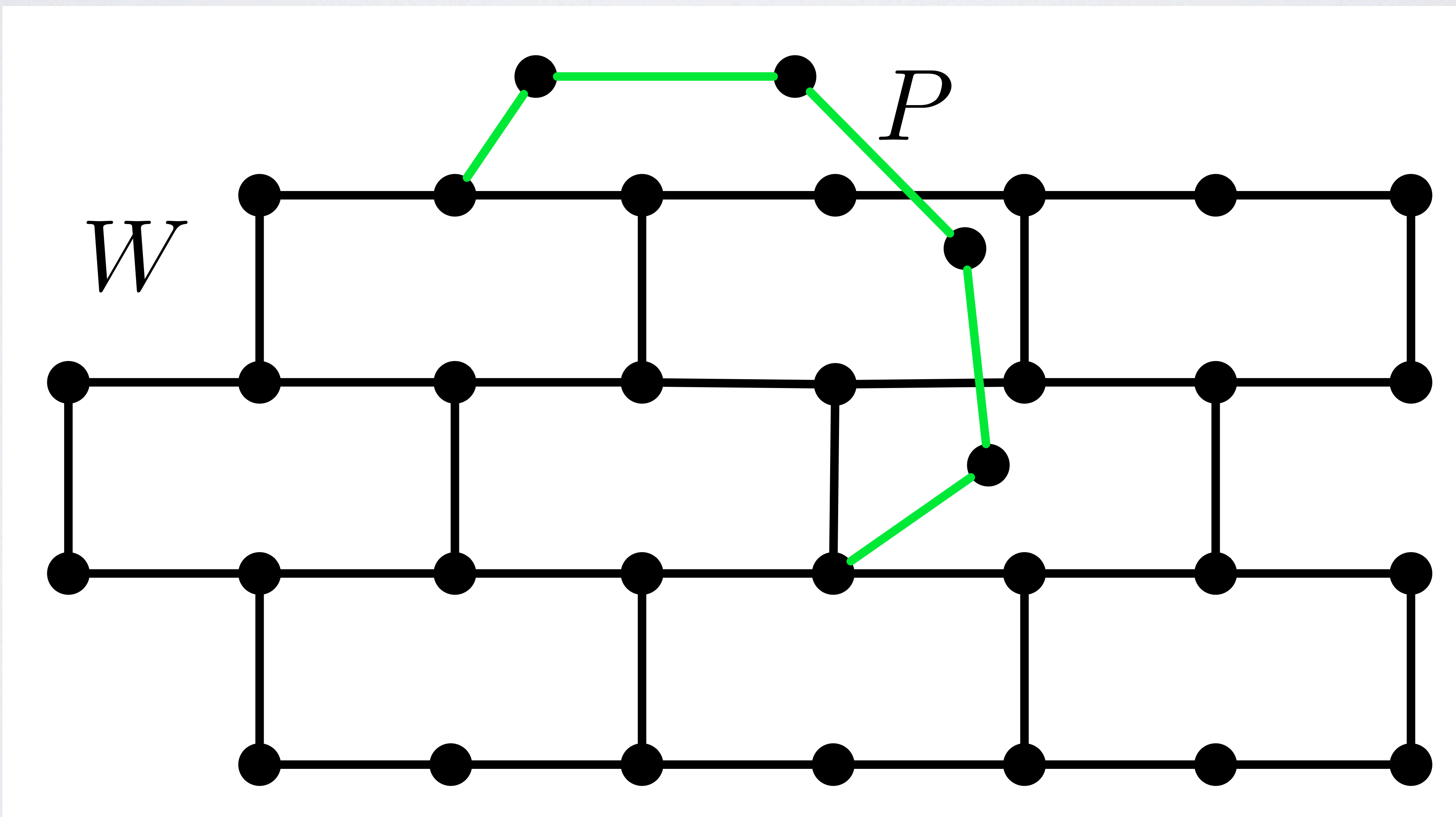
- Basis of the RS algorithm for disjoint paths and minor testing.
- Inductive base case of KTW proof of the full Graph Minor Structure Theorem.
- Recent FPT algorithm for testing subdivision containment by Grohe, Kawarabayashi, Marx, W.
- Recent shorter algorithm for finding Graph Minor Decomposition of Grohe, Kawarabayashi, Reed.

# 3 Main Tools of the Proof:

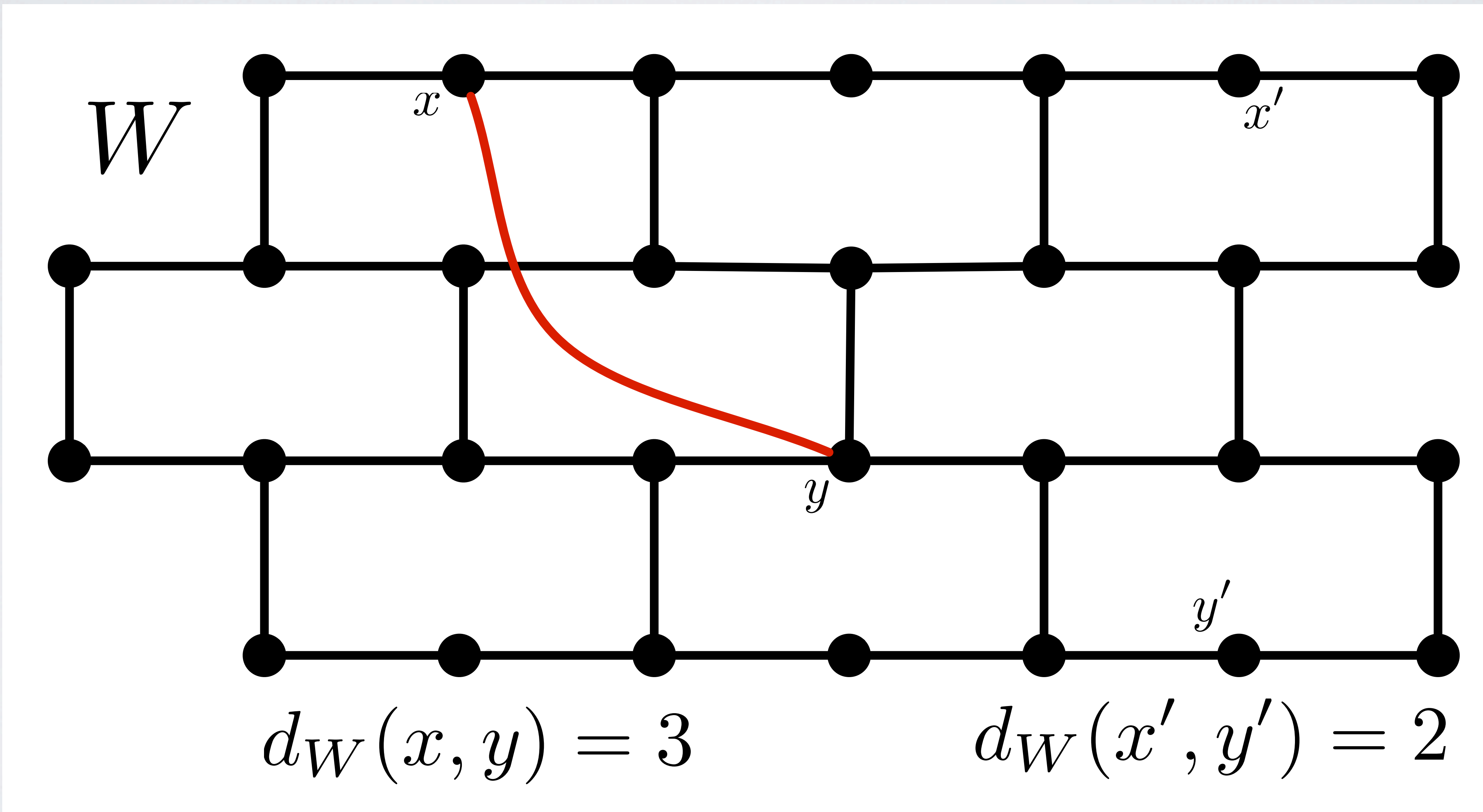
- Lemma on disjoint W-paths with endpoints pairwise far apart.
- Theorem of RS for the 2-disjoint paths problem.
- Principle: if something happens often enough, then it happens many times in the same way.

# 1. Disjoint W-paths

H a subgraph of G, then an **H-path** is a path P with ends in V(H), no internal vertex in H, and E(P) ∩ E(H) = ∅.



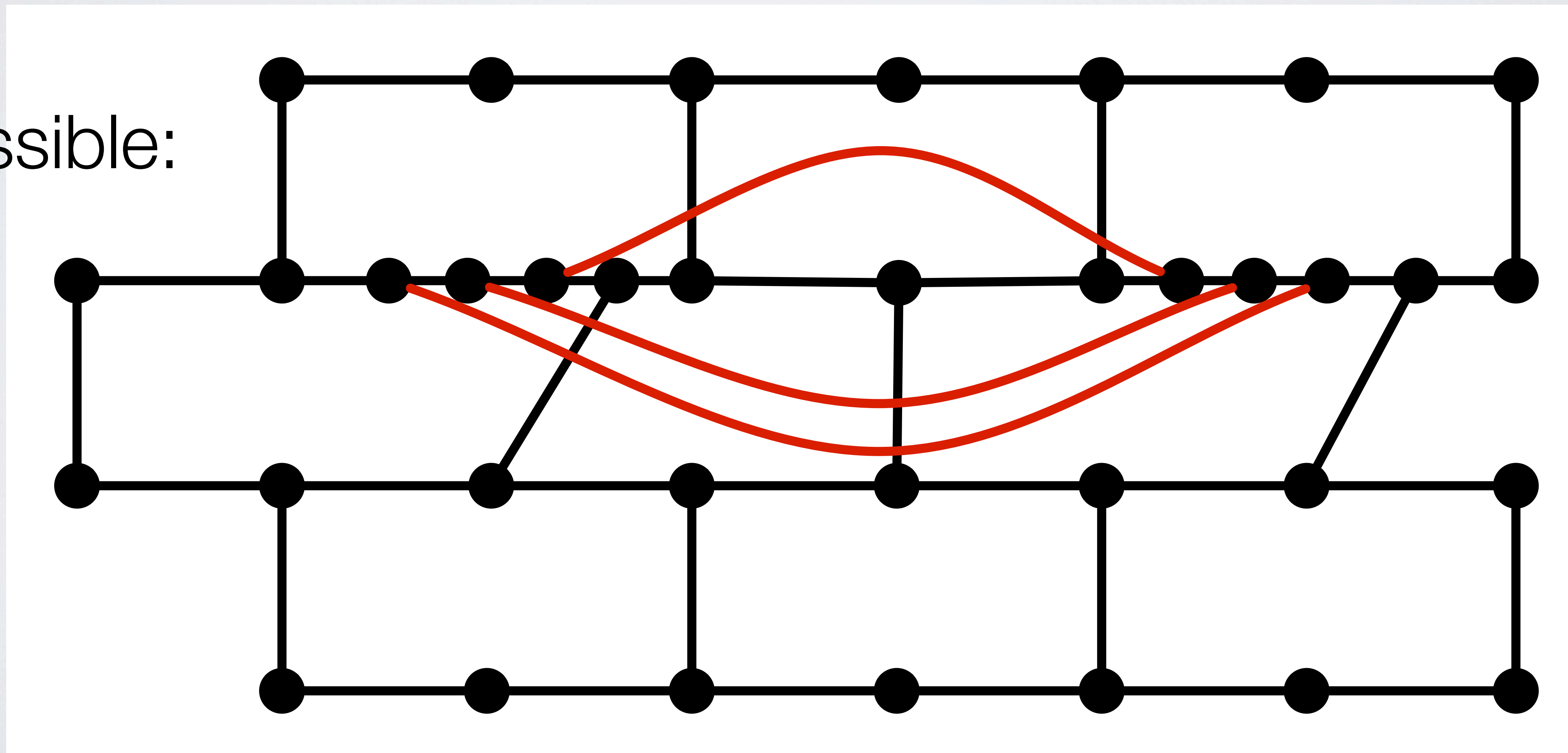
W a wall, x,y ∊ V(W), let **$d_W(x,y)$** be the minimum number of times a curve in the plane from x to y intersects W.



Disjoint W-paths $P_1,...,P_k$ are **t-semi-dispersed** if we can label the ends of $P_i$ as $x_i$ and $y_i$ such that: $d_W(x_i,y_i) \geq t$ and $d_W(y_j,y_i) \geq t$.

We would like an EP-result for t-semi-dispersed paths...

Impossible:

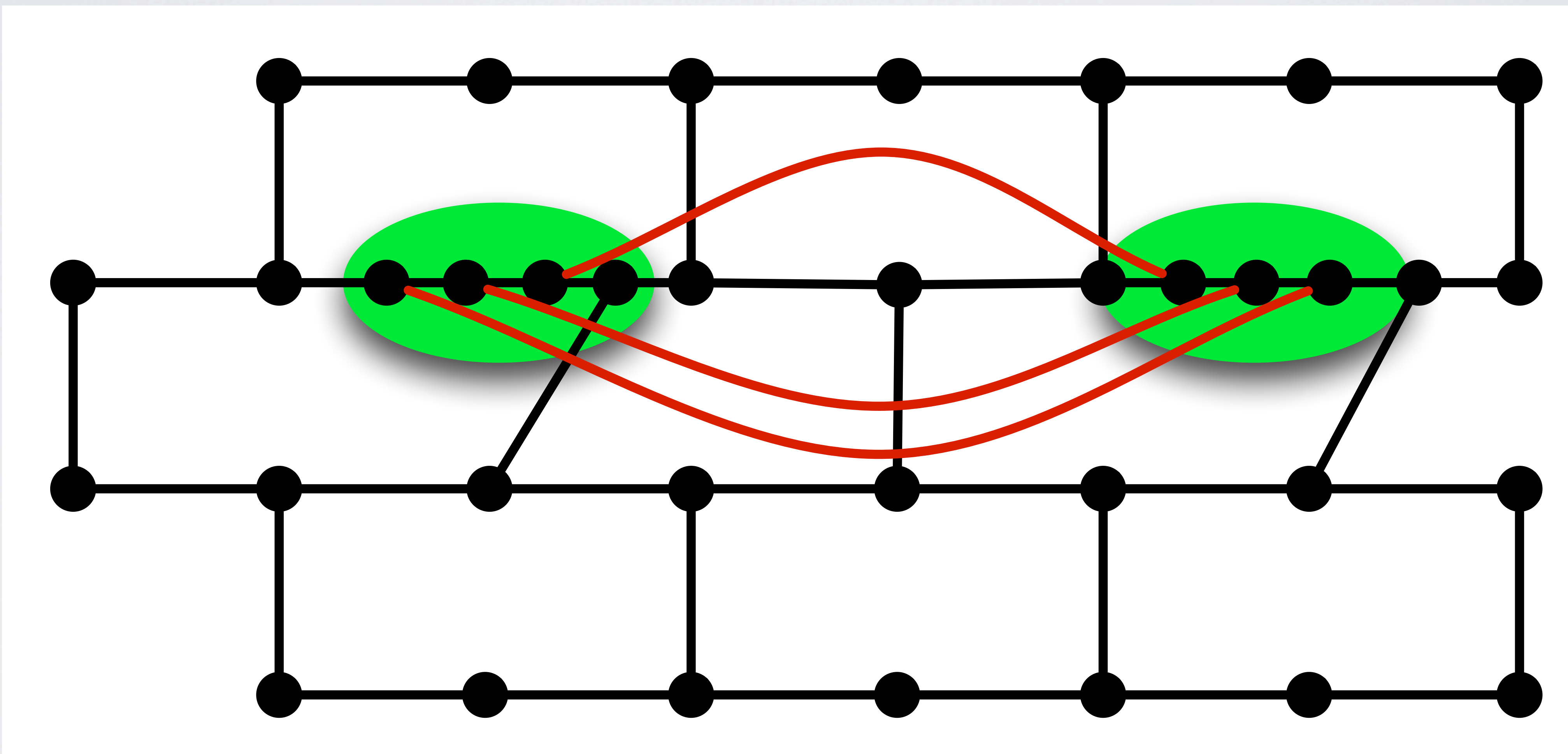


Given W, $t \geq 1$, $x \in V(W)$, let **$B_t(x)$** $= \{y: d_W(x,y) \leq t\}$

Lemma: Given W and $t,k \geq 1$, either there exist disjoint W-paths $P_1,\ldots,P_k$ which are t-semi-dispersed, or there exists a set X, $|X| \leq k-1$ and $Z \subseteq V(W)$, $|Z| \leq 3k-3$, such that every W-path P with ends x and y, either has $d_W(x,y) \leq t$, $V(P) \cap X \neq \varnothing$,

or both x and y are contained in $\bigcup_{z \in Z} B_t(z)$

Even if no bounded set hits all long paths, two balls cover the ends of every long path.



# 2. RS theorem for 2-disjoint paths

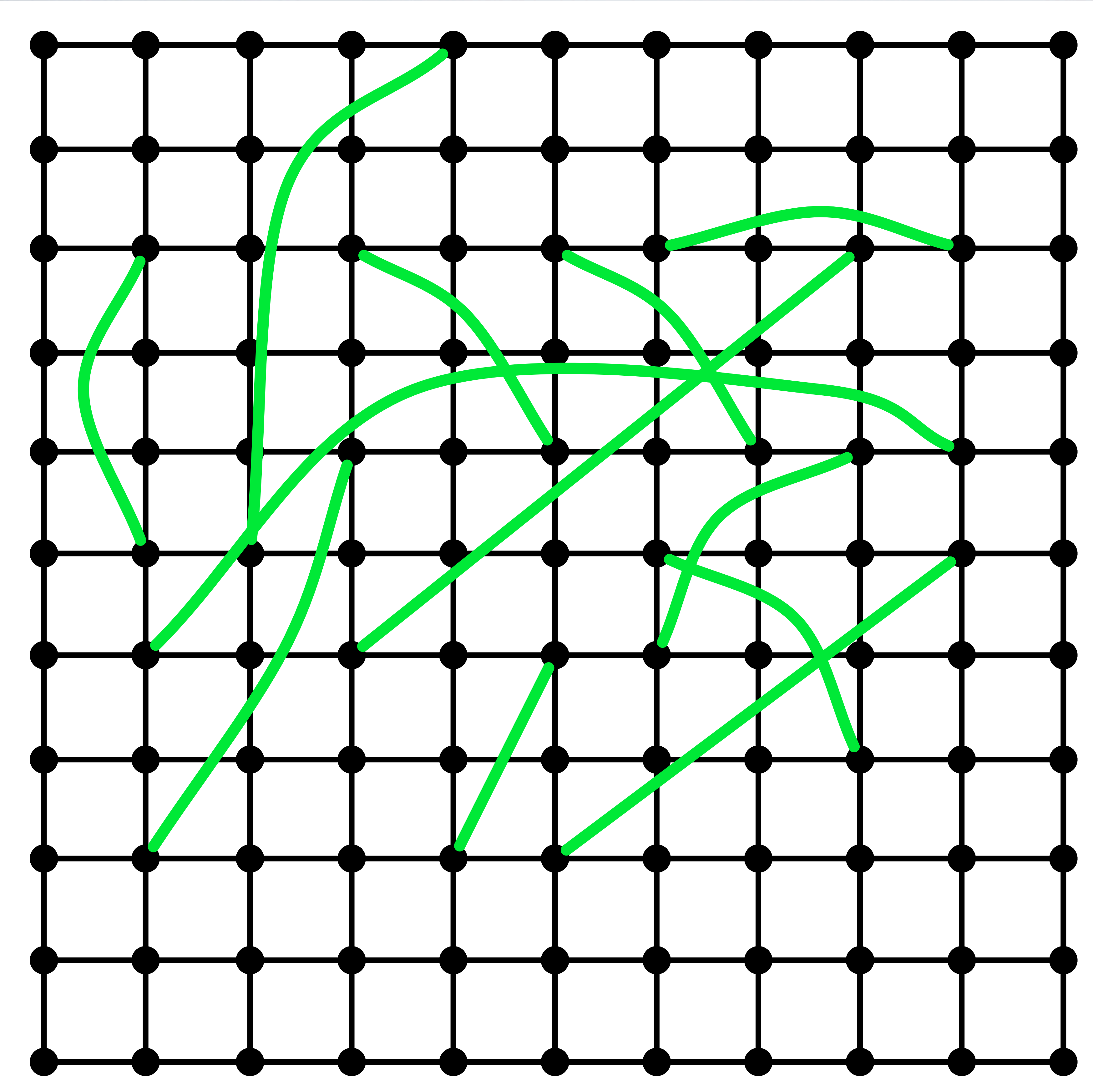
Theorem (RS): G a graph and $s_1,s_2,t_1,t_2$ vertices, then there exist disjoint paths $P_1,P_2$ such that the ends of $P_i$ are $s_i$ and $t_i$ if and only if G is not C-flat where C is the cycle on vertices $s_1,s_2,t_1,t_2$.

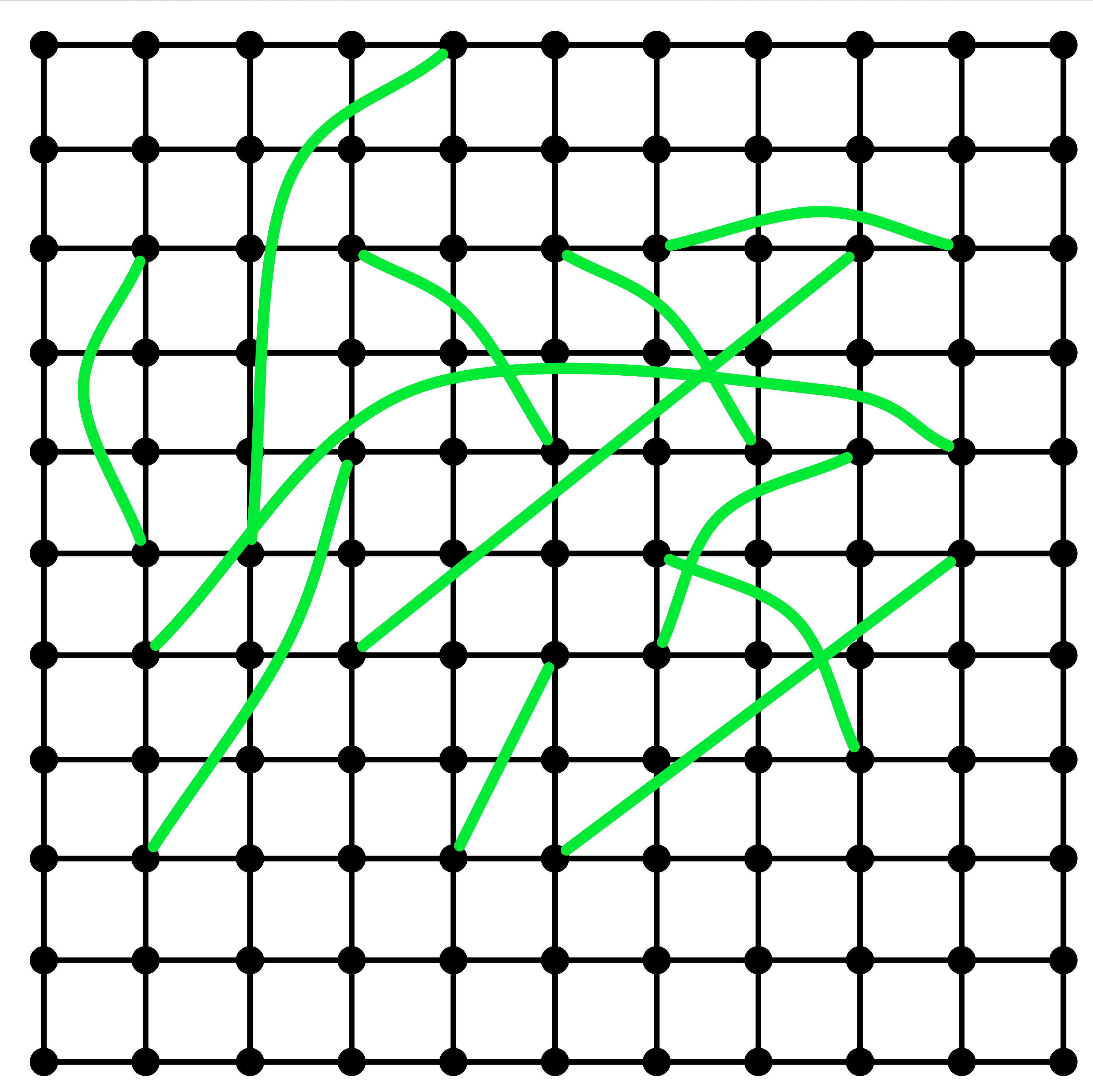
# 3. If something happens often enough, it happens many times in the same way

Theorem: Let $t,k \geq [x_1,y_1],\ldots,[x_k,y_k]$ be k intervals on the real line. If $k \geq t^2$, then either there exist t pairwise disjoint intervals, or there exists z contained in t distinct intervals.

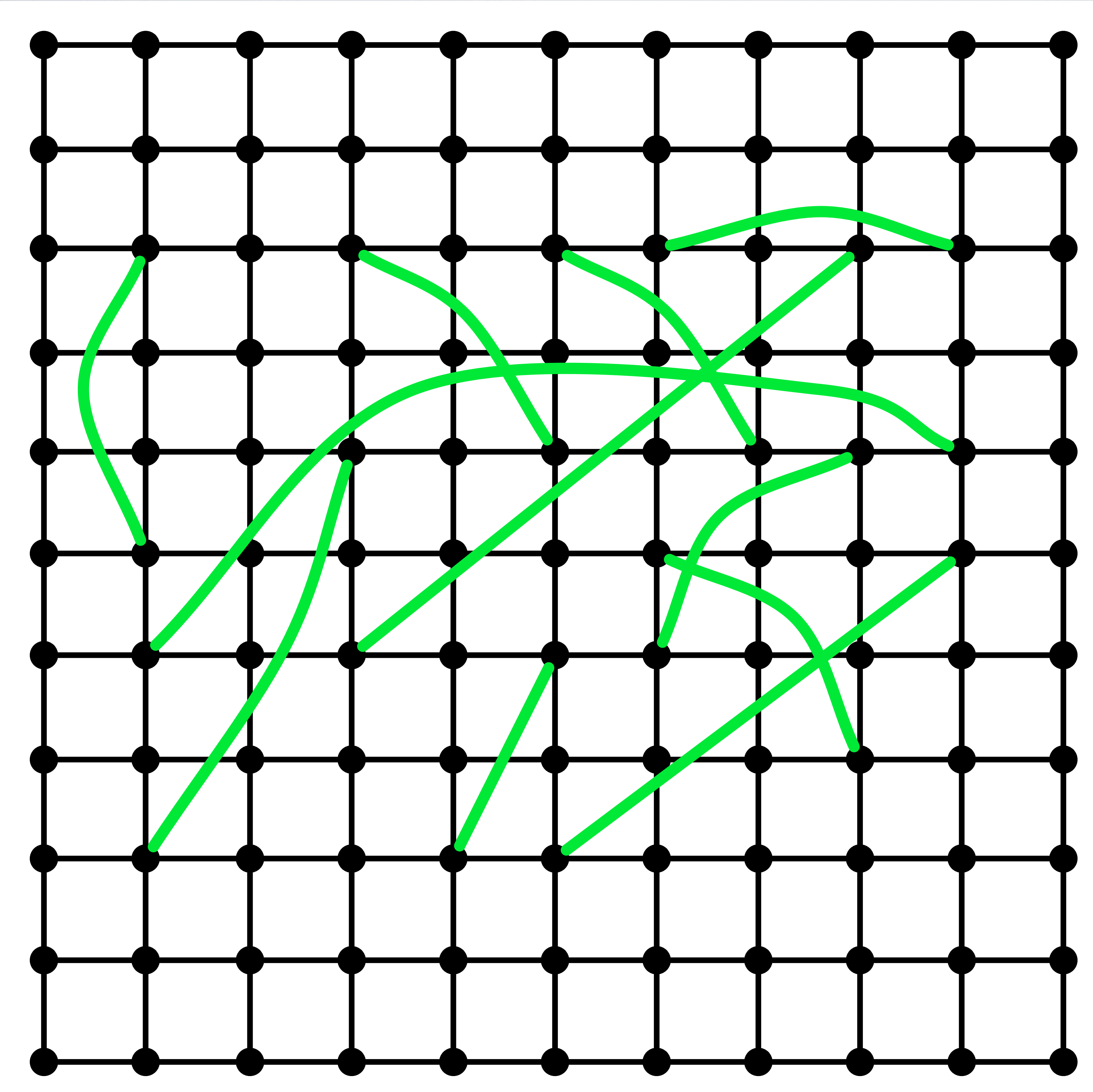
Theorem: There exists a polynomial function f such that if $P_1,\ldots,P_k$, $k \geq f(t)$, are pairwise disjoint f(t)-semi-dispersed W-paths on a wall W, then $W \cup P_1 \cup \ldots \cup P_k$ contains $K_t$ as a minor.

Easier statement: There exists a polynomial functions f satisfying the following. Let $P_1,\ldots,P_k$, $k \geq f(t)$, be pairwise disjoint W-paths with endpoints contained in a set $X \subseteq V(W)$ for a grid W. If the vertices of X are pairwise at distance f(t), then $W \cup P_1 \cup \ldots \cup P_k$ contains $K_t$ as a minor.





- ∃ at most one endpoint within dist $t^2$ of the boundary. Delete it.



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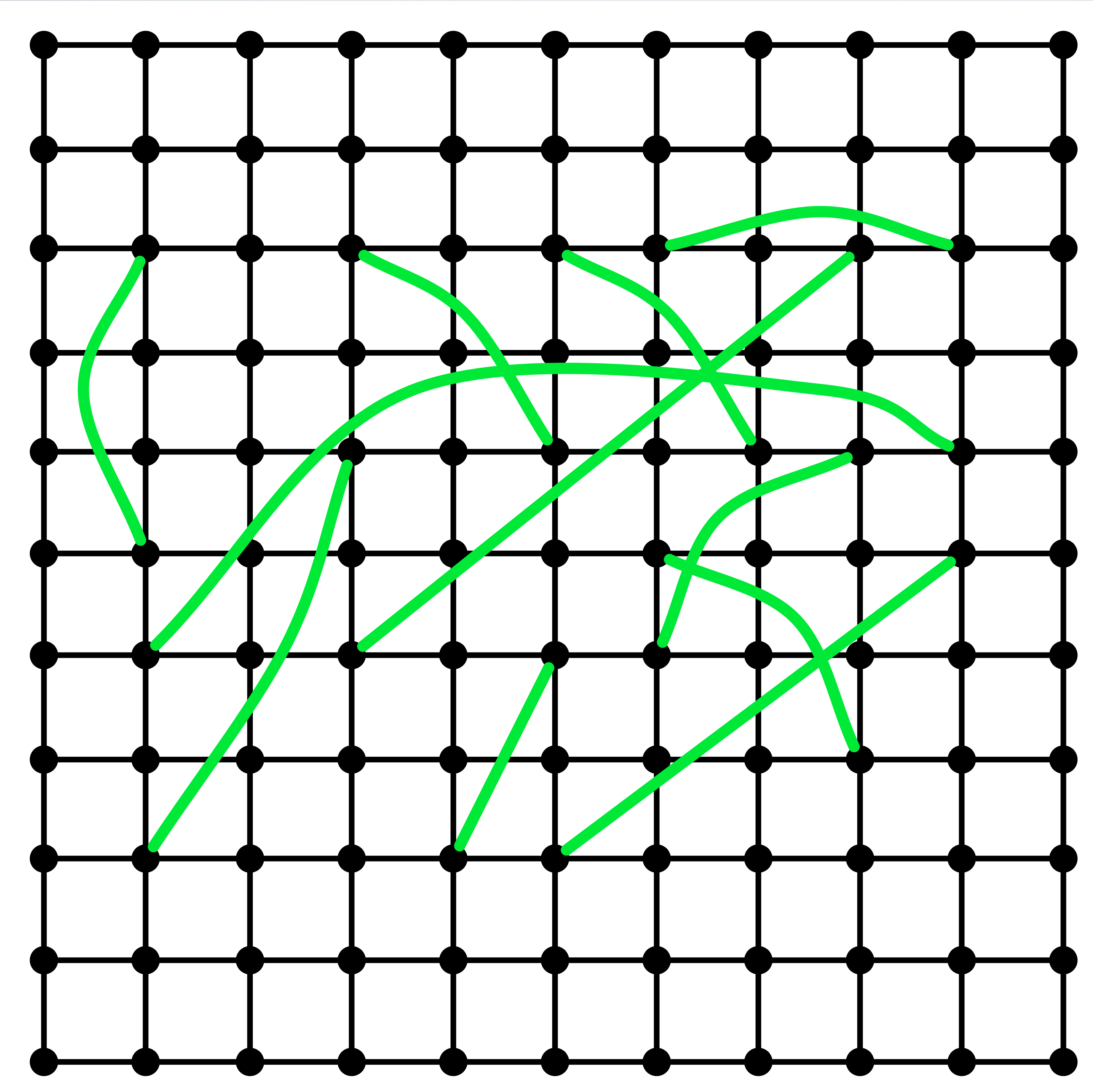
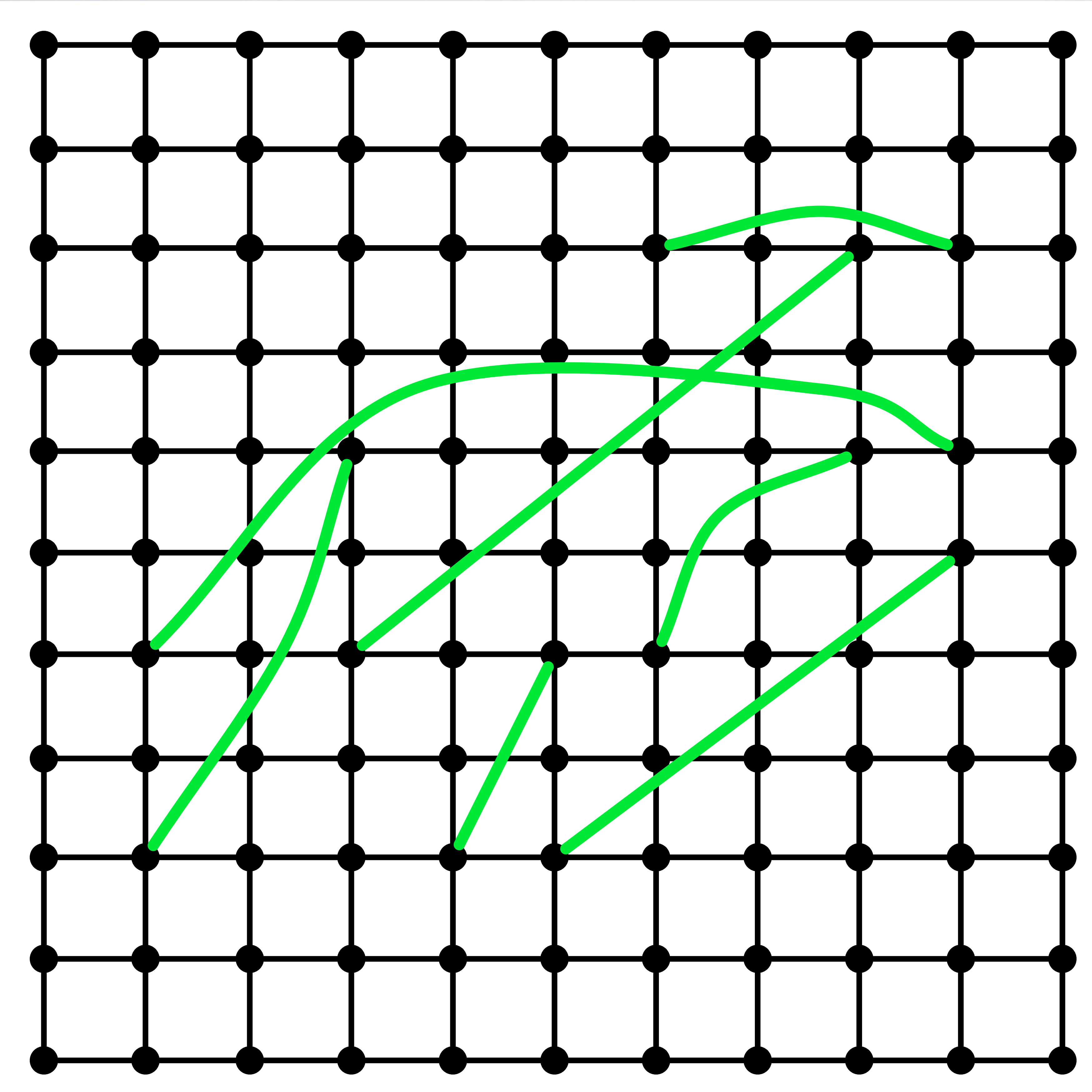


- ∃ at most one endpoint within dist $t^2$ of the boundary. Delete it.
- At least half the line segs have slope m with $0 \leq m < \infty$.



- $\exists$ at most one endpoint within dist $t^2$ of the boundary. Delete it.
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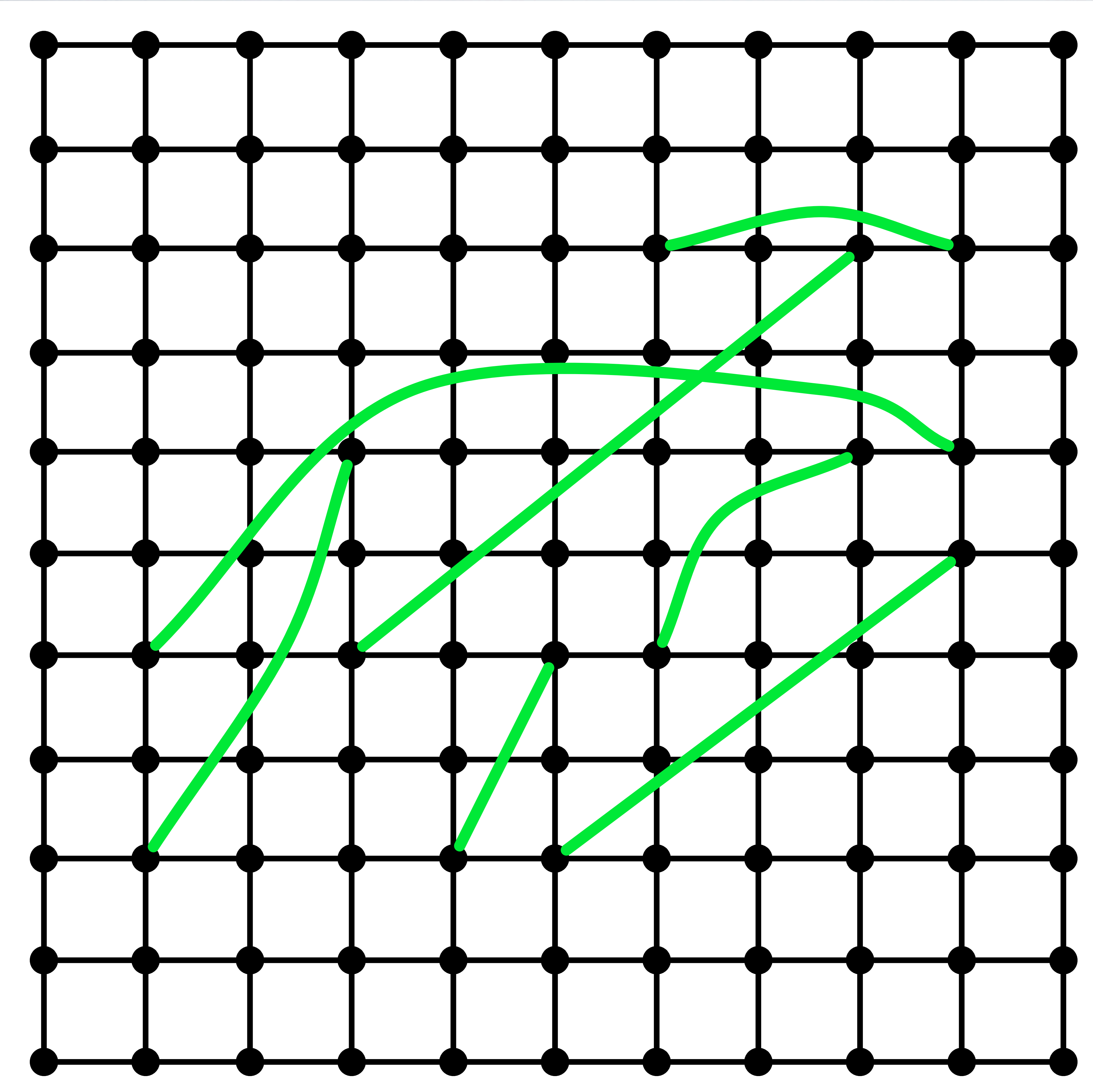

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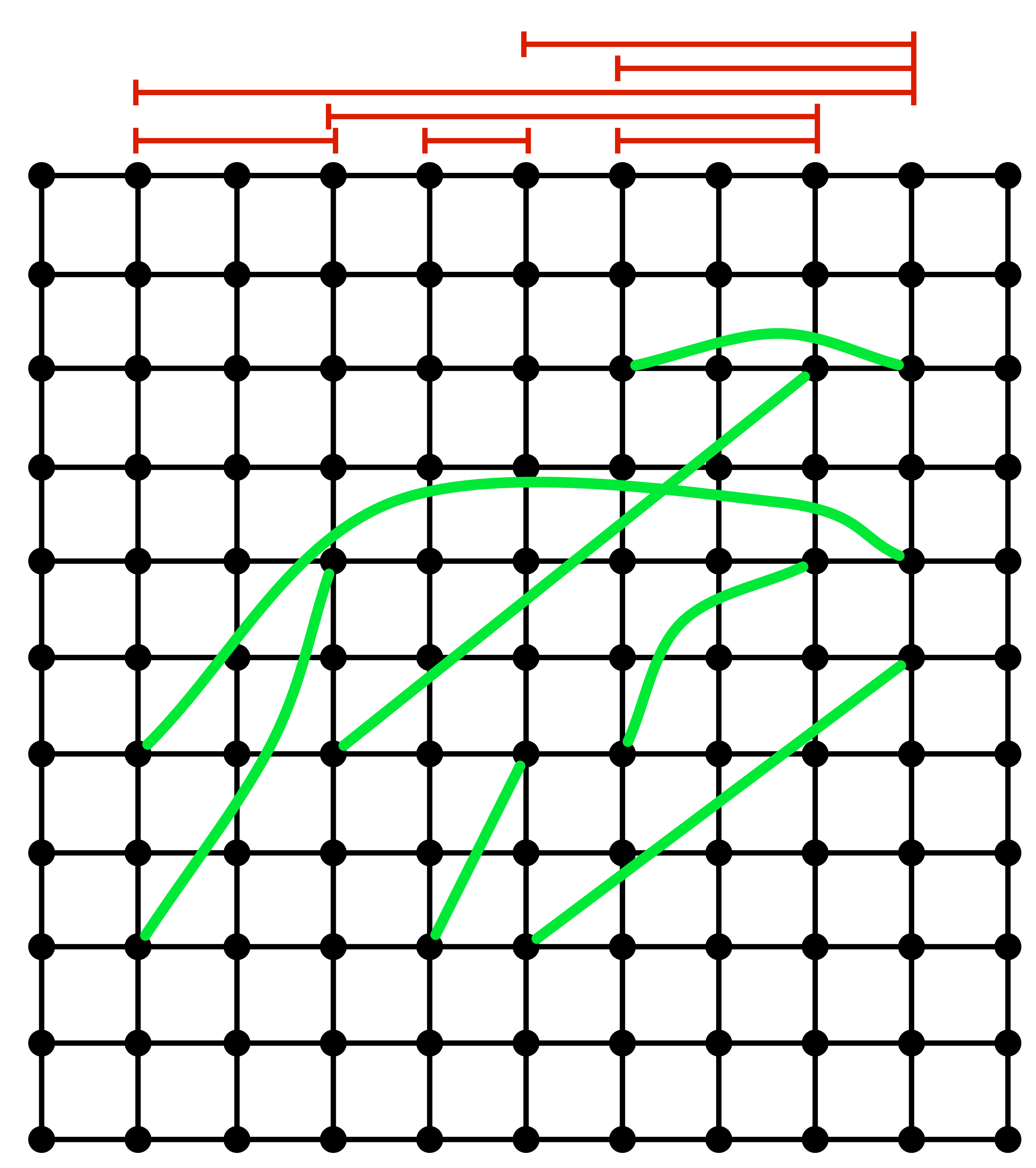
- Apply interval lemma to x-axis intervals of the segments.



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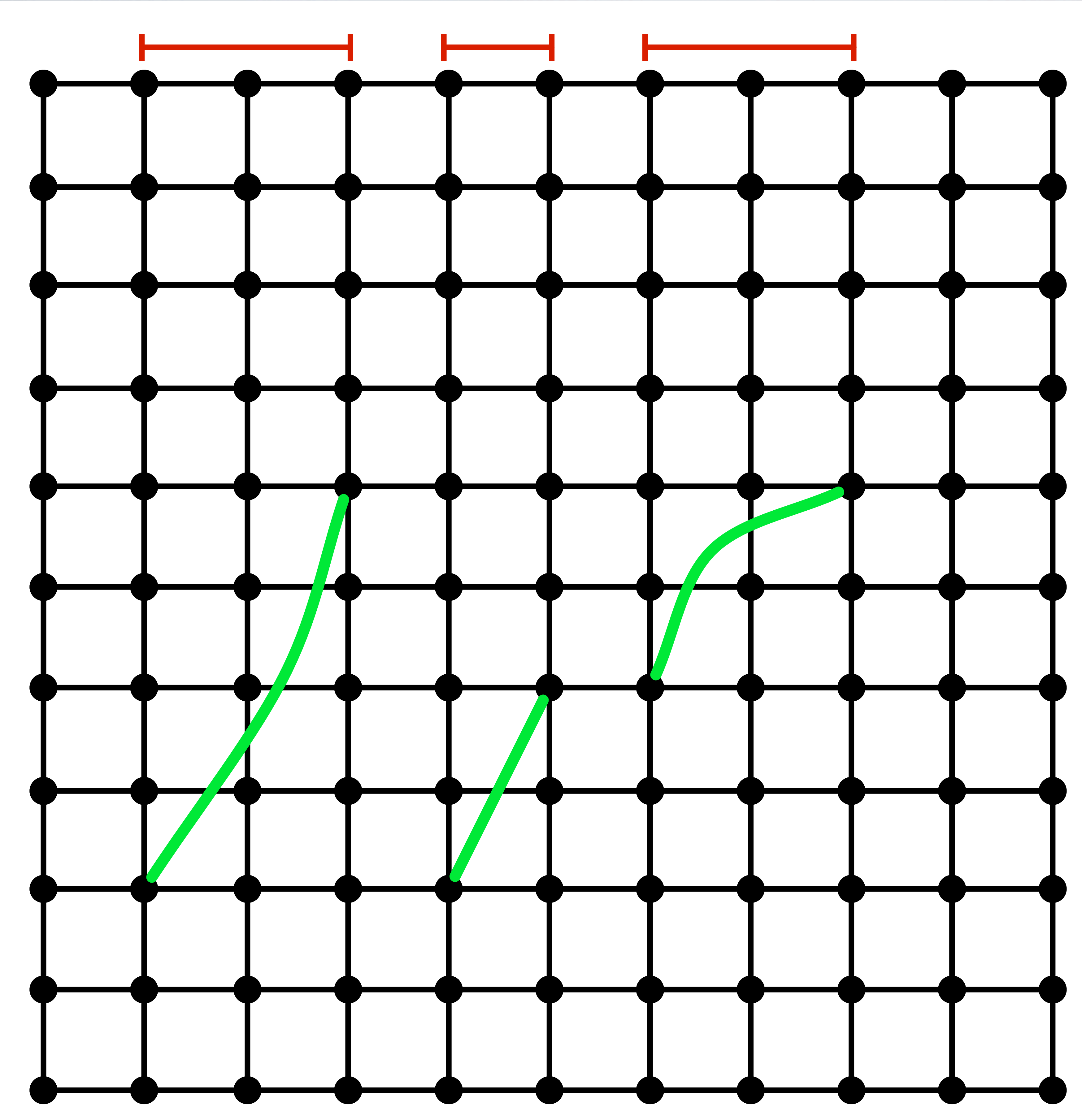

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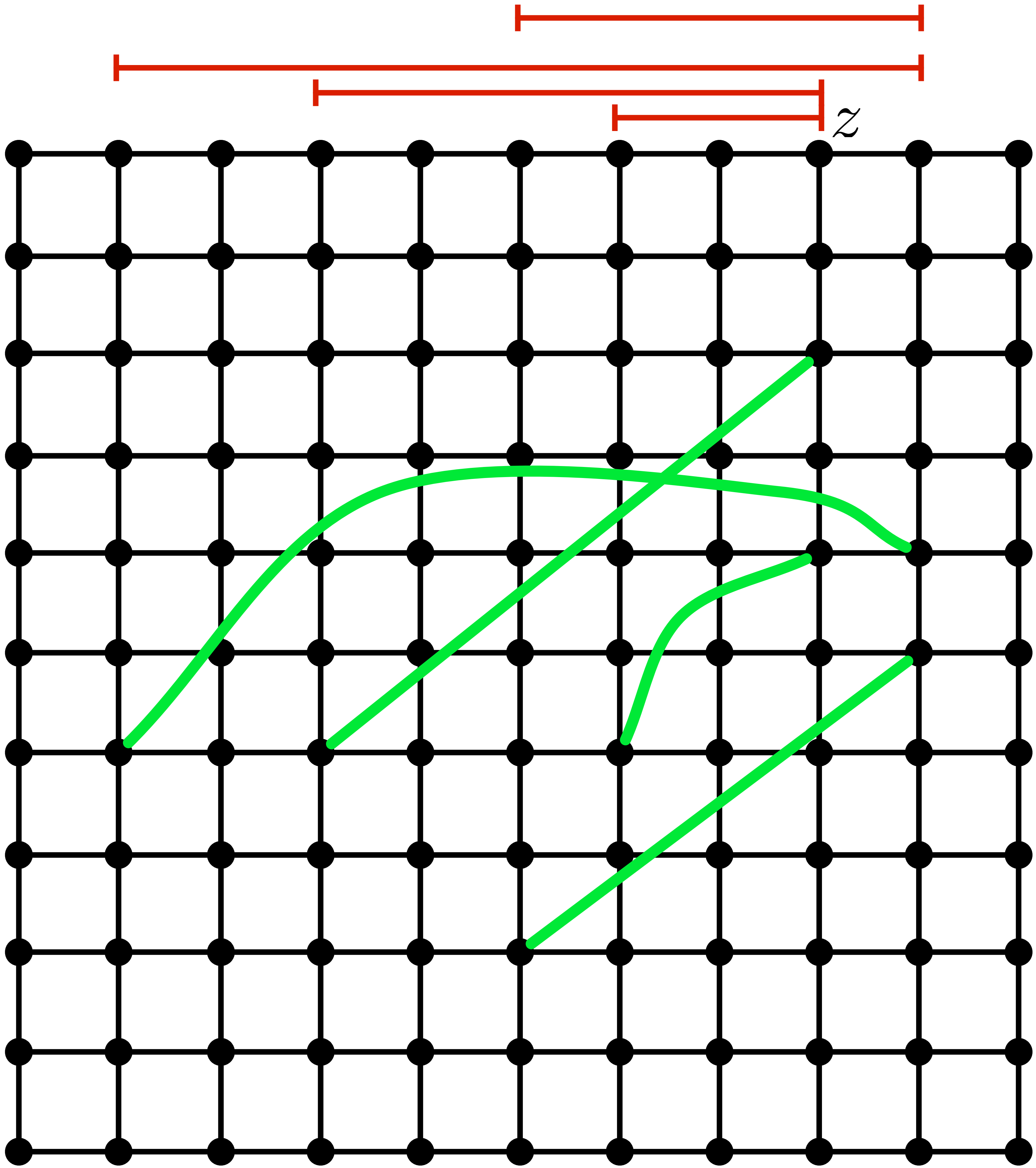
- If we have $t^2$ disjoint intervals, we find as a minor the $2t^2$ grid with crosses in the middle face.

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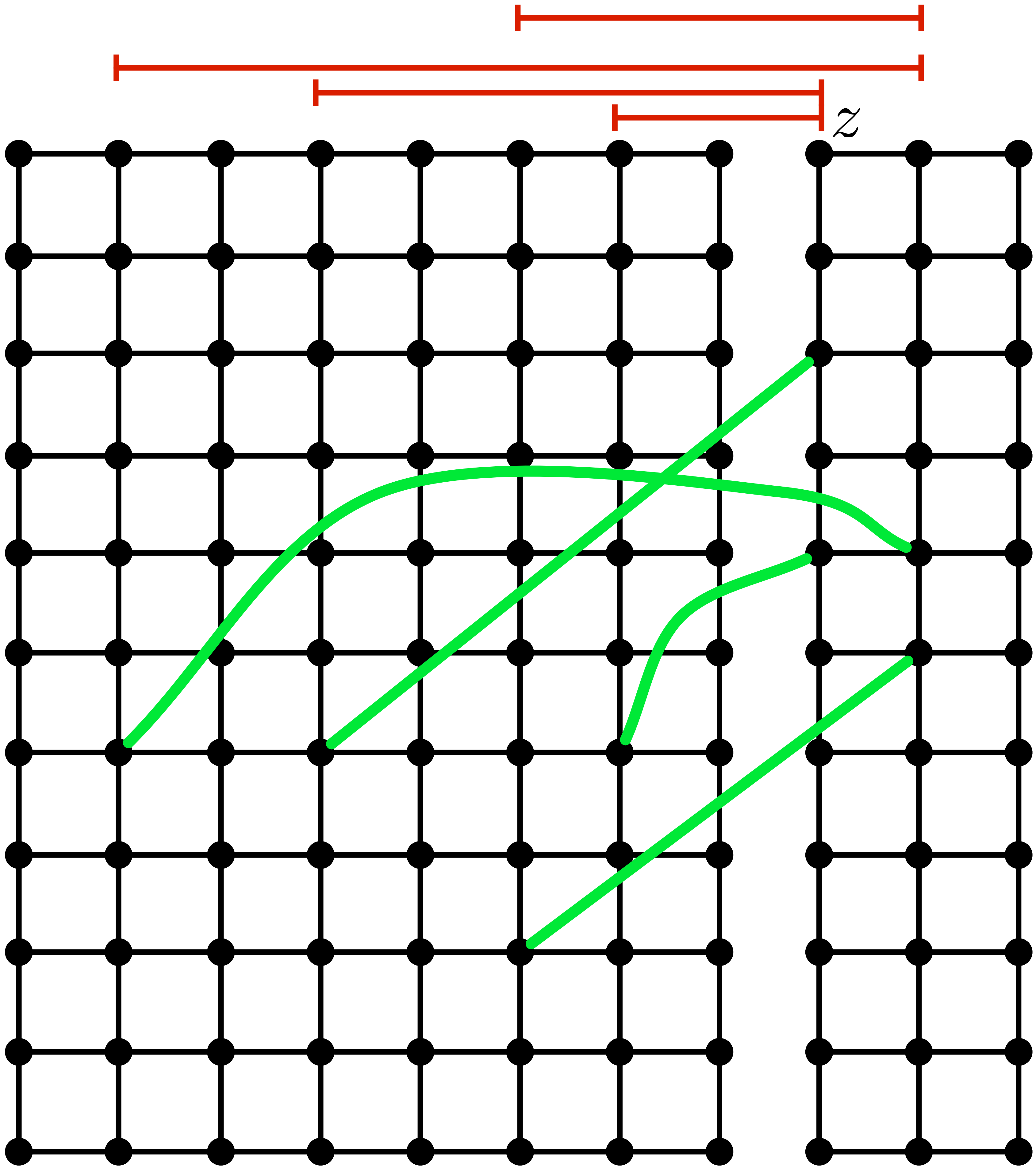
- Apply interval lemma to x-axis intervals of the segments.
- If we have $t^2$ disjoint intervals, we find as a minor the $2t^2$ grid with crosses in the middle face.
- Otherwise, we have many intervals containing a common point z. Split the grid into two halves along the x=z vertical line.

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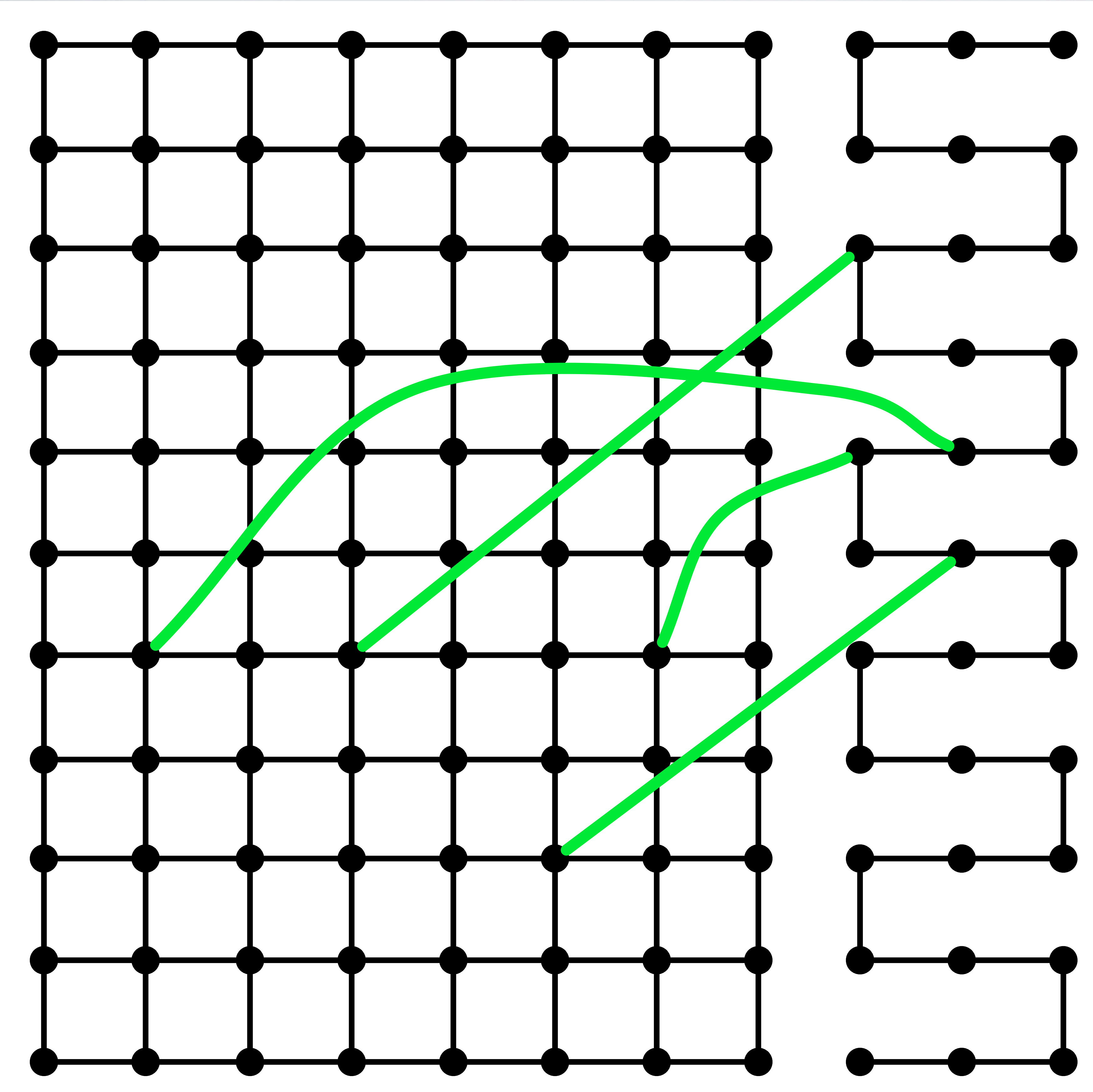

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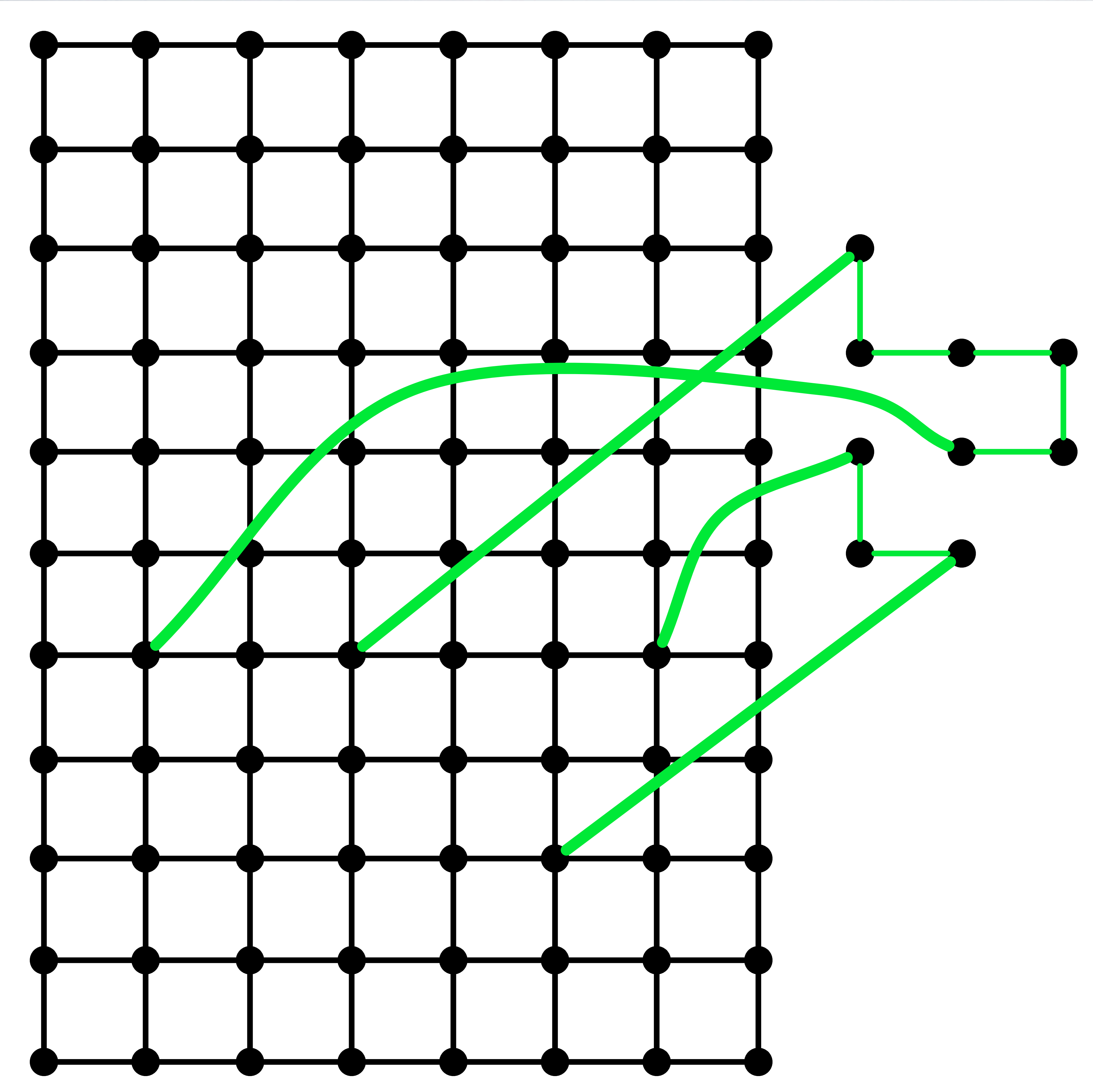

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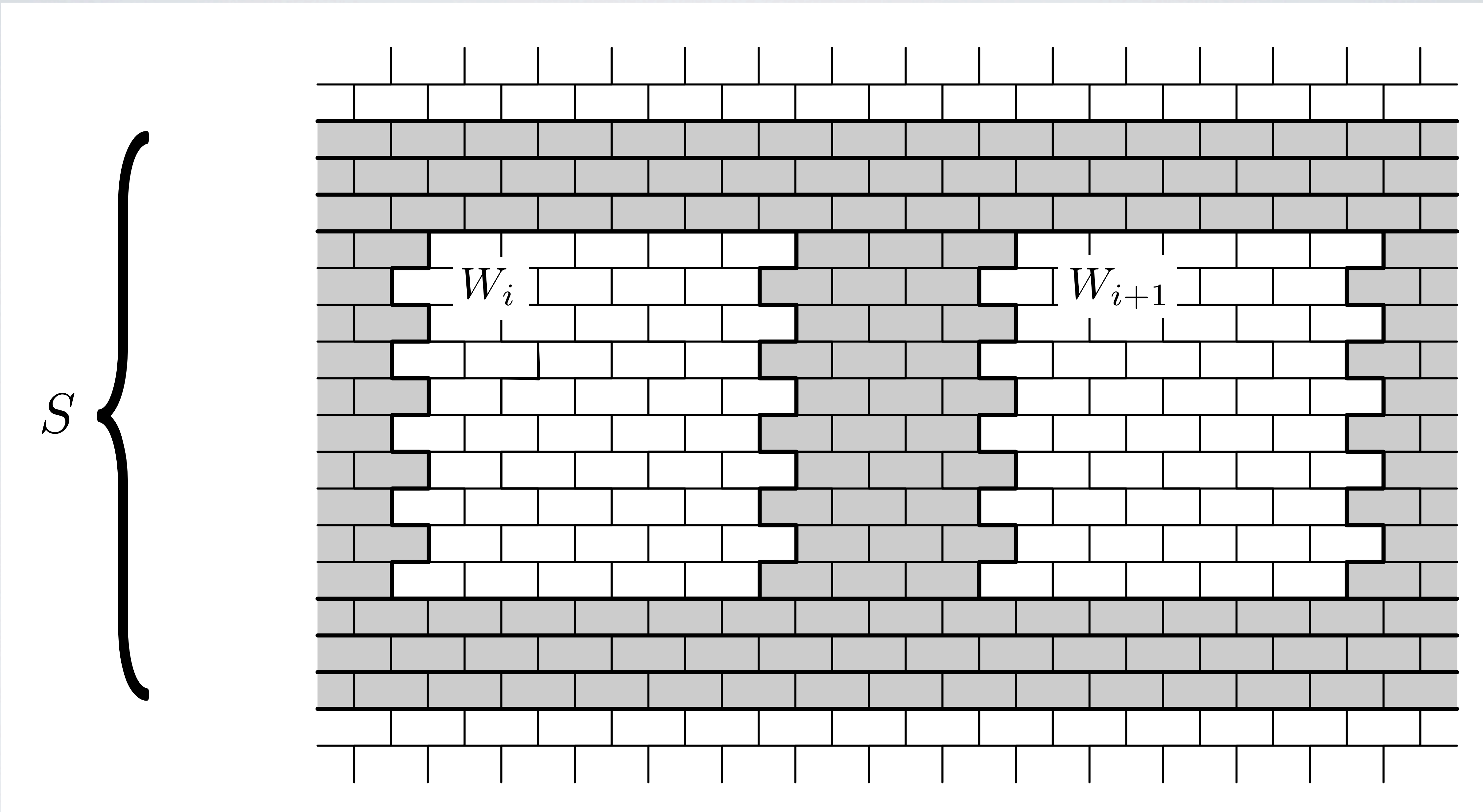
# Outline of the proof of the Weak Structure Theorem

1. Apply the semi-dispersed paths lemma to the wall.

2. If there exist $f_1(t)$-semi-dispersed set of $f_2(t)$ disjoint W-paths, then we find the $K_t$ minor for appropriately chosen polynomials $f_1$ and $f_2$.

3. Otherwise, there exists a bounded set X and bounded number of balls $B_1,\ldots,B_k$ hitting all long W-paths.

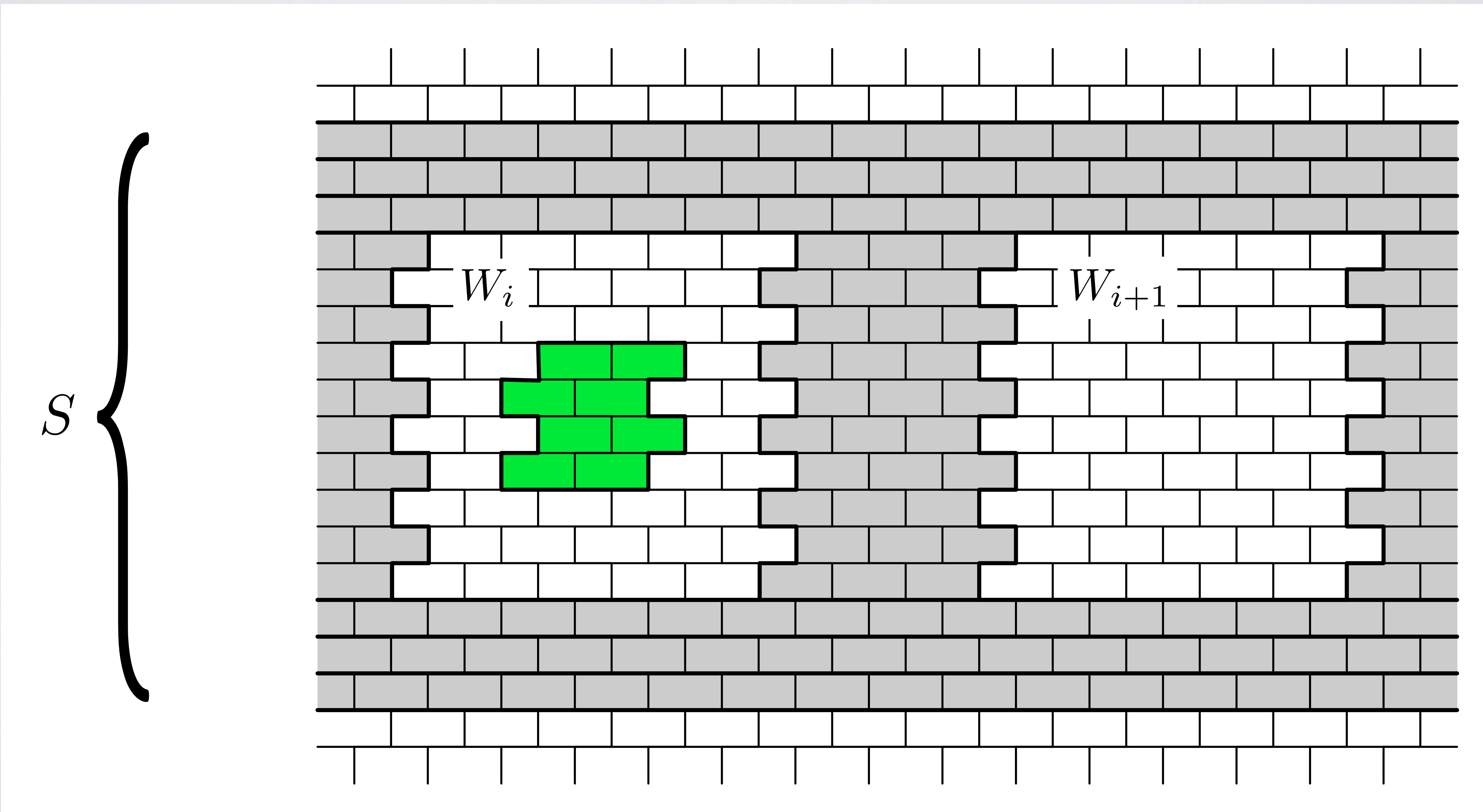
Thus, there exists a wide horizontal strip S of the wall avoiding all the balls $B_1,\ldots,B_k$ as well as X.




Let $W_i$ be (r+poly(t))-walls in the center of the strip spaced poly(t) apart.

Apply the 2-paths theorem to each $W_i$ (and the bridges attaching to $W_i$) along with the four corners.

If some $H_i$ does not have the desired paths, we find a smaller flat subwall inside.



# An algorithm

Theorem: There exists an algorithm with
**Input**: a graph G on n vertices and m edges, r, $t \geq 1$, and a R-wall W, $R = 50000t^{24}(24t^2 + r)$
**Output**: either a $K_t$ minor grasped by W or a set A, $|A| \leq 12288t^{24}$ and a nontrivially flat r-subwall of W' with $V(W') \cap A = \emptyset$.
**Runtime**: $O(t^{24}m + g(n,m))$ where g(n,m) is the runtime for the 2-disjoint paths algorithm.

The 2 disjoint paths problem:

- RS showed a O(nm) time algorithm.
- Kapadia, Li, Reed announced O(m) but unpublished.