When are directed graphs Well-quasi-ordered

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Graphs are well-quasi-ordered under a containment relation “<“ if for every infinite sequence:

$G_1, G_2, G_3, G_4, G_5, G_6, G_7, \ldots.$

there exists i, j such that $G_i < G_j$

Conjecture (Wagner): Graphs are well-quasi-ordered under taking minors.

Theorem (Robertson-Seymour 04): Wagner’s conjecture is true.

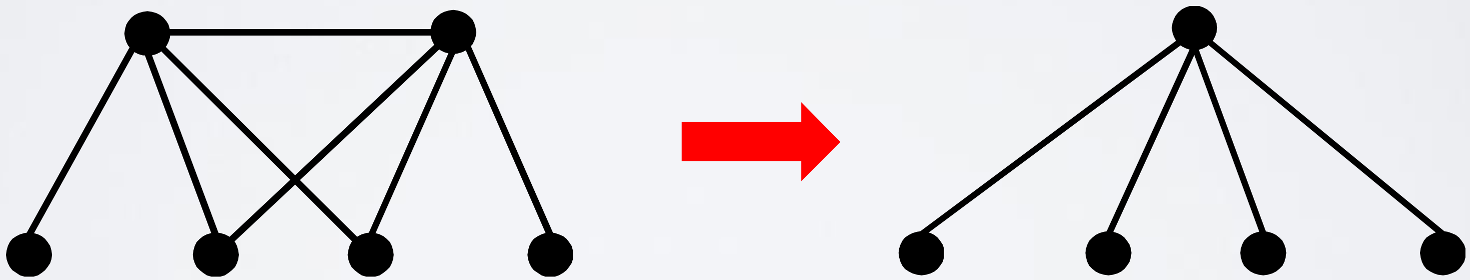
Proof builds a deep and general theory about graph minors

Question: Does a similar theory of directed minors exist?

The graph G contains H as a minor if H can be obtained from G by:

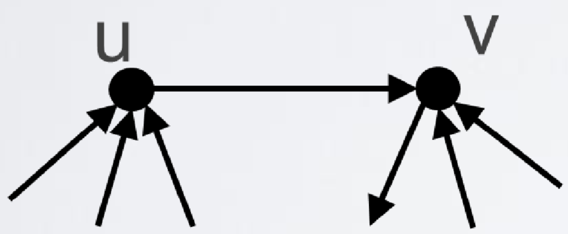
– Deleting edges and isolated vertices and

– Contracting edges (deleting parallel edges that arise).

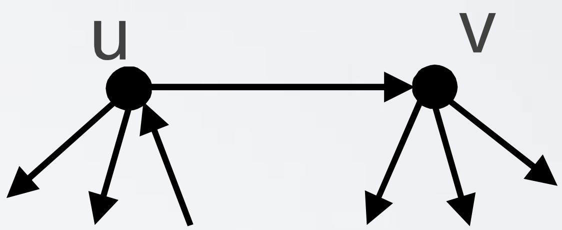


The digraph G contains digraph H as a minor if H can be obtained from G by:

- Deleting edges and isolated vertices and
- Contracting edges uv such that:
    a. u has only one out-edge or
    b. v has only one in edge.



Case a.



Case b.

Known as butterfly-minor - we will only consider butterfly minors of directed graphs.

Little is true for digraph minors:

(RS 95) k-disjoint paths problem is polytime solvable for fixed k

(FHW 80) Directed 2-disjoint paths problem is NP-complete

(RS 95, GKMW 11) Subdivision testing is polytime solvable for fixed H

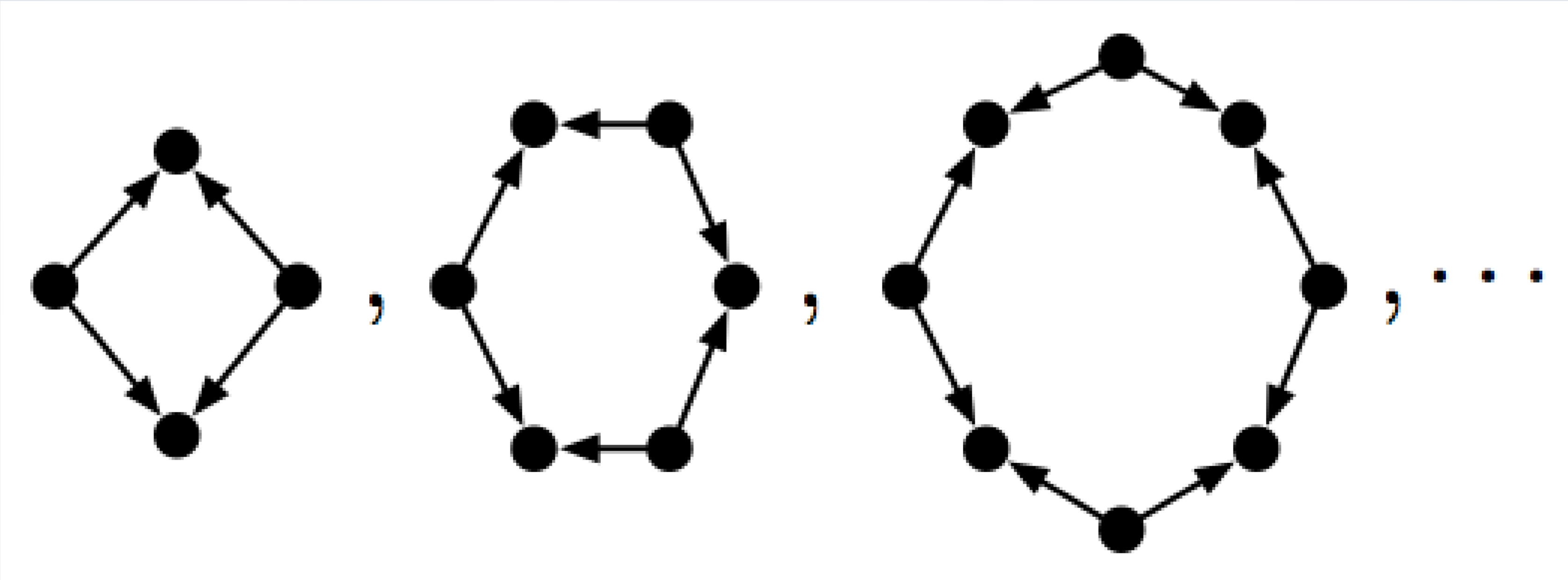
Directed subdivision testing is NP-complete for fixed H

(RS 03) There exists a structure theorem for excluded minors.

Likely that no similar theorem exists for digraphs

(RS 05) Graphs are well-quasi-ordered under minors.

Digraphs are not well-quasi-ordered:



The alternating cycles of length 2k, k ≥ 1 form a counterexample.

An alternate possibility: Graphs are not well-quasi-ordered under topological minors.

Theorem: (Kruskal 1960) Trees are well-quasi-ordered under topological minors.

Question: Can we identify classes of digraphs which are well-quasi-ordered under taking minors?

Def: A (minor) ideal of digraphs is a set of digraphs closed under taking minors.

Theorem (CMOSW 14^+): A minor ideal **F** of digraphs is well-quasi-ordered if and only if there exists N such that for all n > N, **F** does not contain:

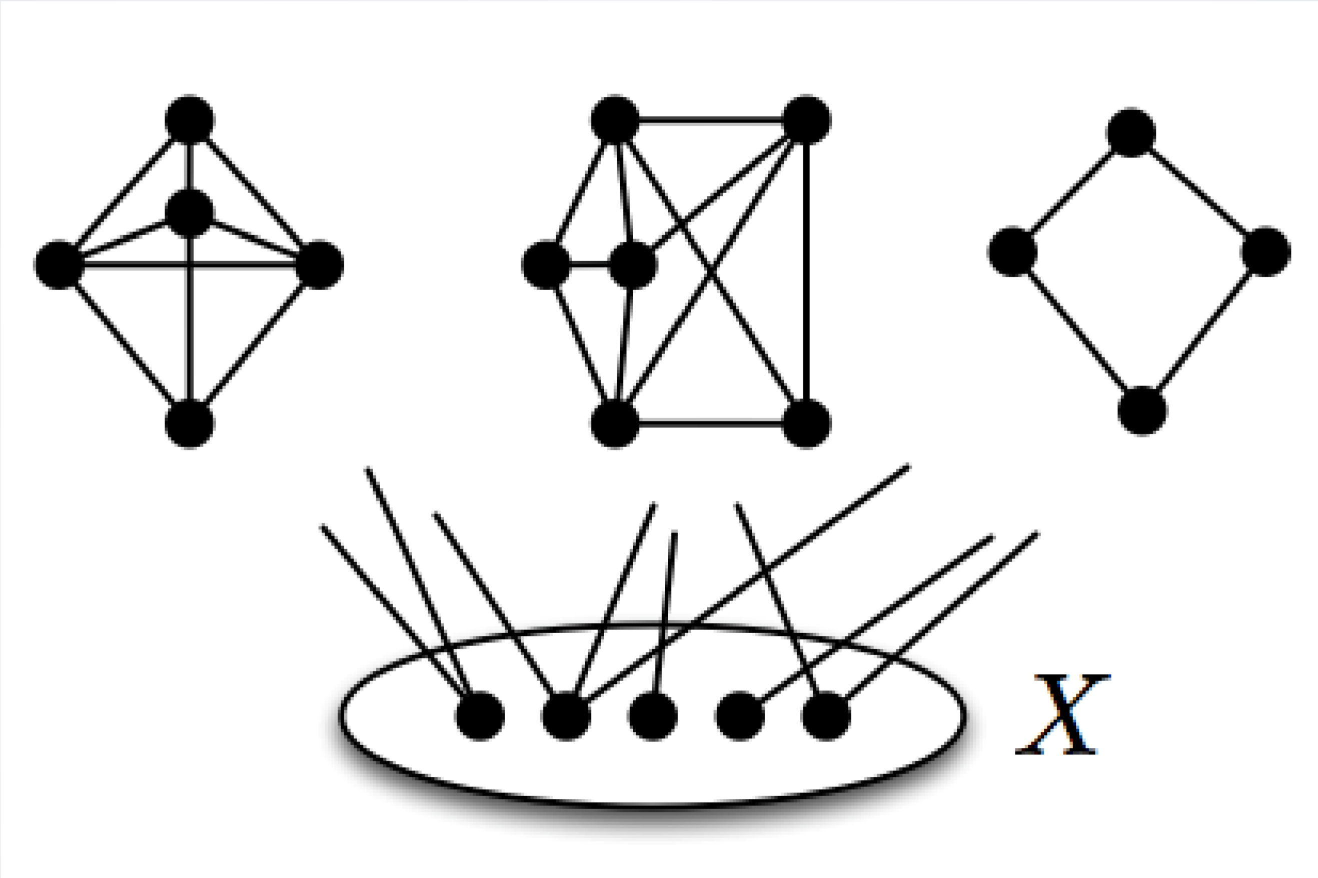
- an alternating cycle of length 2n, or
- an alternating rooted path of length n

Theorem: (Ding, 92) Let **F** be a subgraph ideal such that there exists an integer k such that **F** does not contain the path of length k. Then **F** is well quasi ordered.

Proof outline: proceed by induction on k.

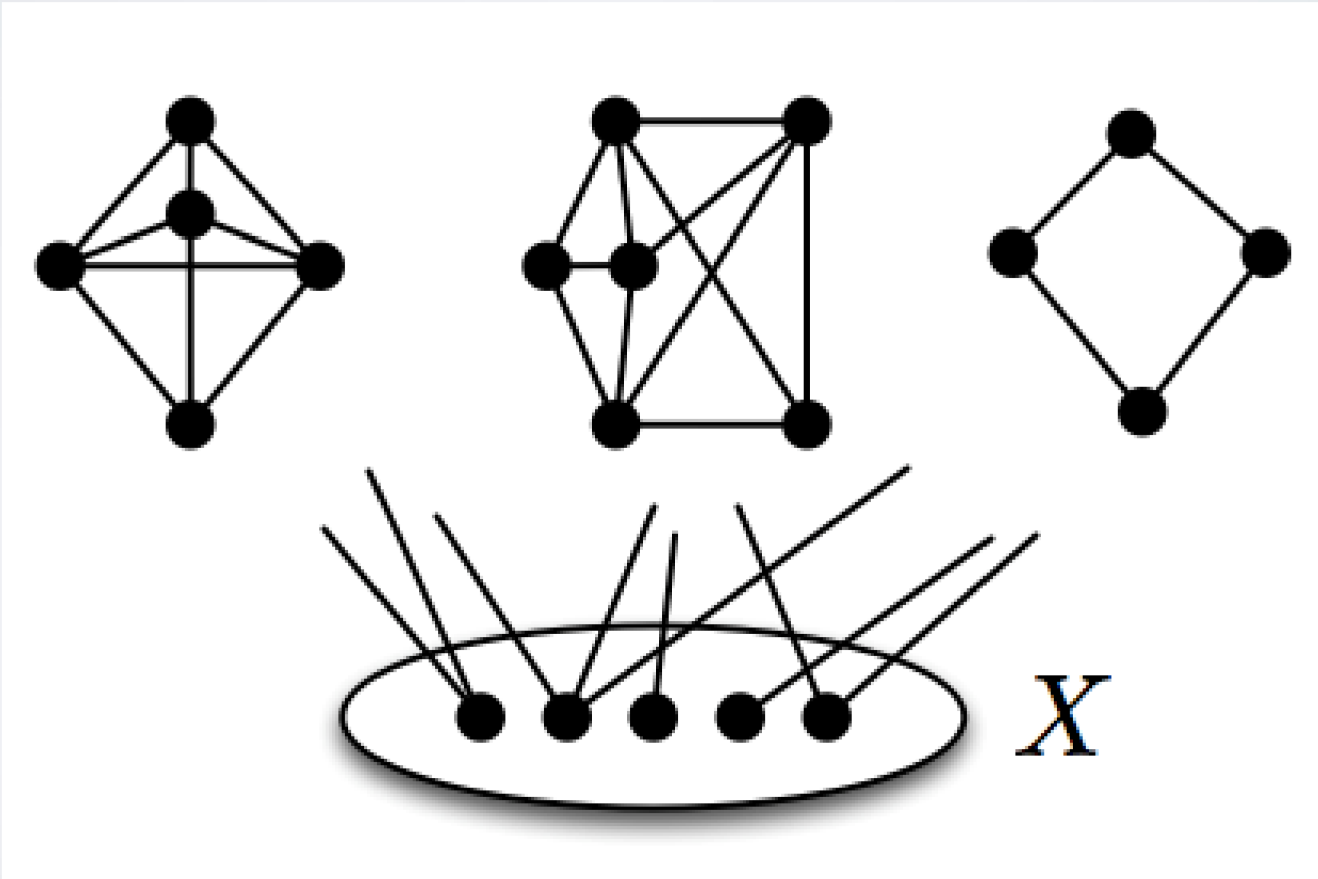
Proposition: G a connected graph that does not contain a path of length k. There exists $X \subseteq V(G)$, $|X| < k$, such that X intersects every path of length k-1.

1. G has no path of length k
2. X a bounded set intersecting all paths of length k
3. Components G_1, G_2, G_3 of G-X have no path of length k-1 - they are well-quasi-ordered by induction.



Strengthen the induction hypothesis:

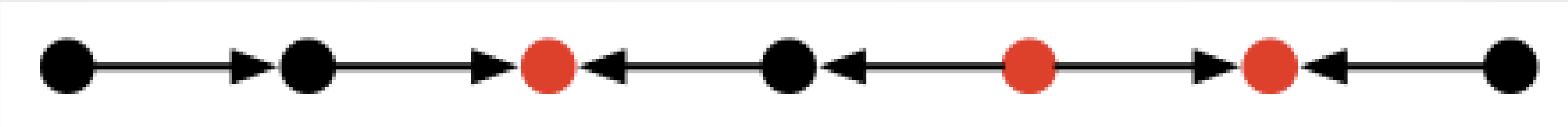
- labeled graphs with no path of length k are well-quasi-ordered under taking labeled subgraphs.



An analog for long paths in directed graphs

Def: G a digraph - let un(G) be the underlying undirected graph.

Def: A k-alternating path is a path in un(G) such there exist exactly k vertices v for which $deg^{in}(v) = 0$ or $deg^{out}(v) = 0$.



an example of 3-alternating path

Theorem: A minor ideal **F** of digraphs is well quasi ordered if and only if there exists N such that for all n > N, **F** does not contain:

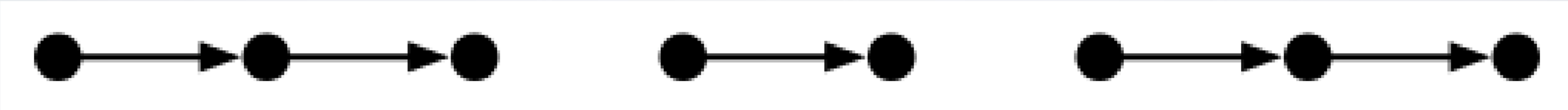
- an alternating cycle of length 2n, or
- an alternating rooted path of length n

Theorem: Let **F** be a minor ideal of digraphs such that **F** does not contain a k-alternating path. Then **F** is well-quasi-ordered.

Proof outline

Induction on k:
base case: k = 1

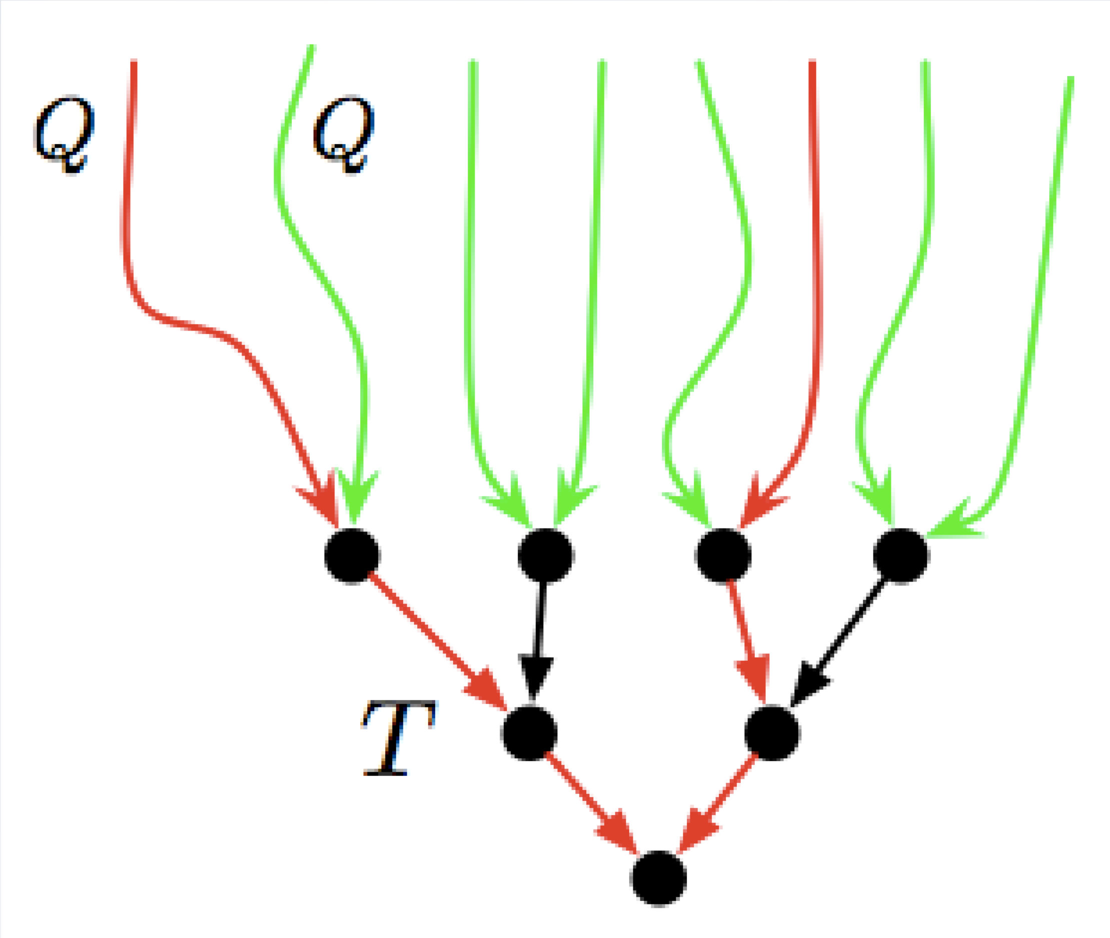
- No vertex has in or outdegree at at least two.
- G is a disjoint union of directed paths. Apply Higgman's lemma.



We would like: no k-alternating path implies bounded number of vertices intersecting all k-1 alternating paths

FALSE

Example 1: Arborescence + alternating paths

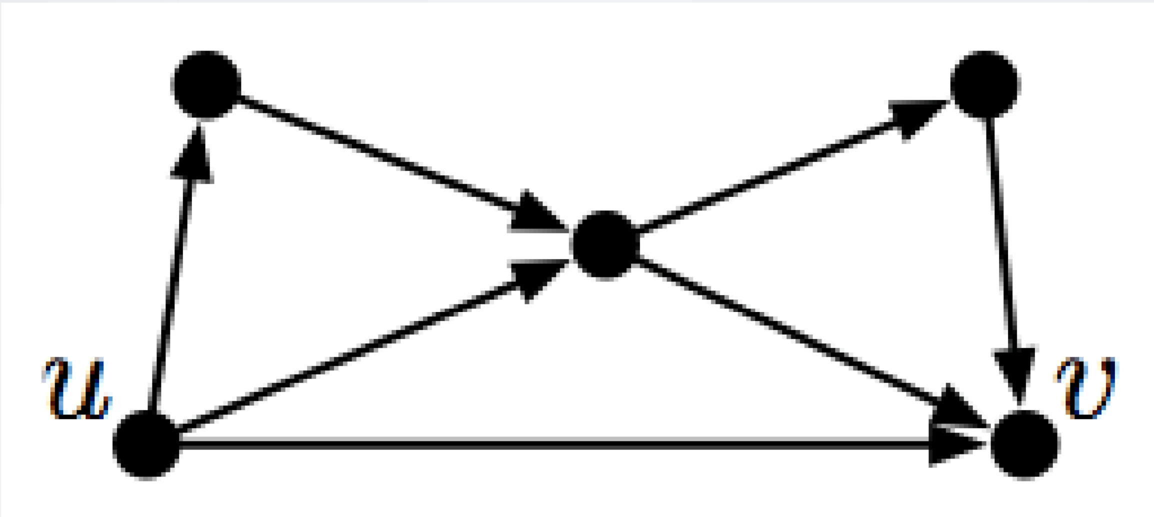


T an arboresence and glue two copies of a k-alternating path Q to each leaf

- Red path is (2k+1)-alternating path
- No (2k+2)-alternating path

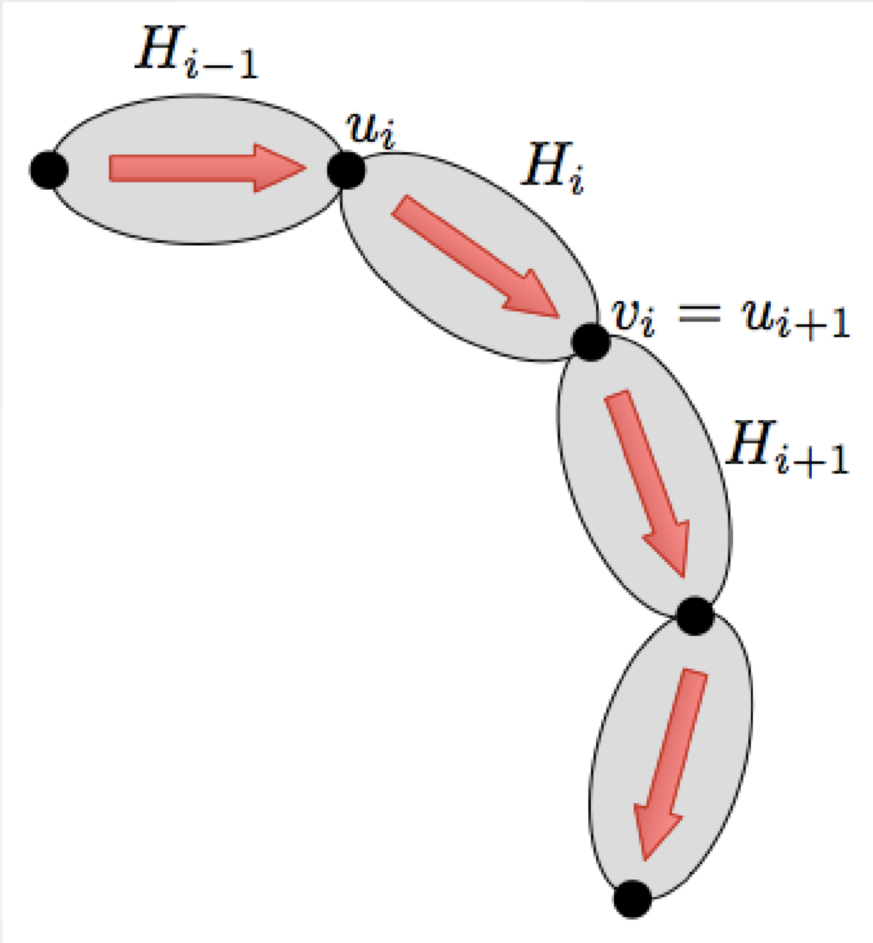
Example 2: String of series-parallel beads

Def: (G, u, v) is a **series parallel triple** if un(G) + uv is a 2-connected graph such that every path in un(G) from u to v is a directed path from u to v in G.



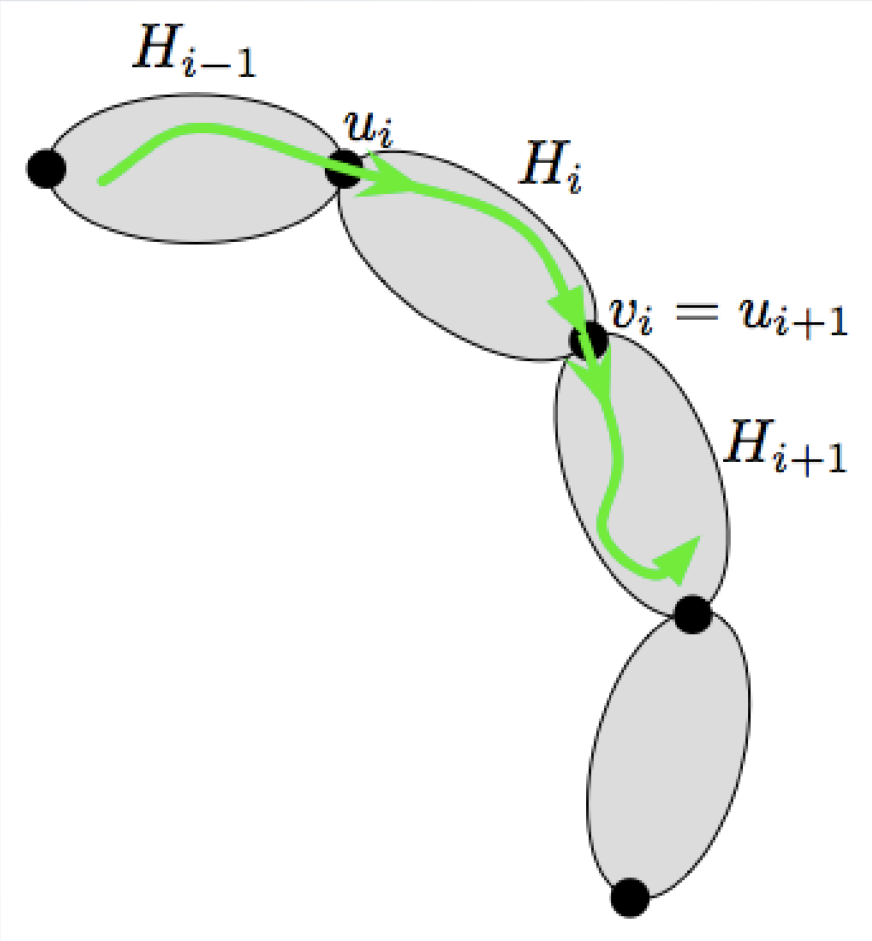
Example 2: String of series-parallel beads

Let (H, u, v) be a series parallel triple, and let (H_i, u_i, v_i) be n isomorphic copies of (H, u, v). Let G be obtained from identifying u_i and v_{i+1} for i = 1,...,n



Example 2: String of series-parallel beads

Prop: If G contains a k-alternating path, then it has a k-alternating path contained in $H_i \cup H_{i+1}$



Let P be a k-alternating path, and pick P to intersect as few H_i as possible.

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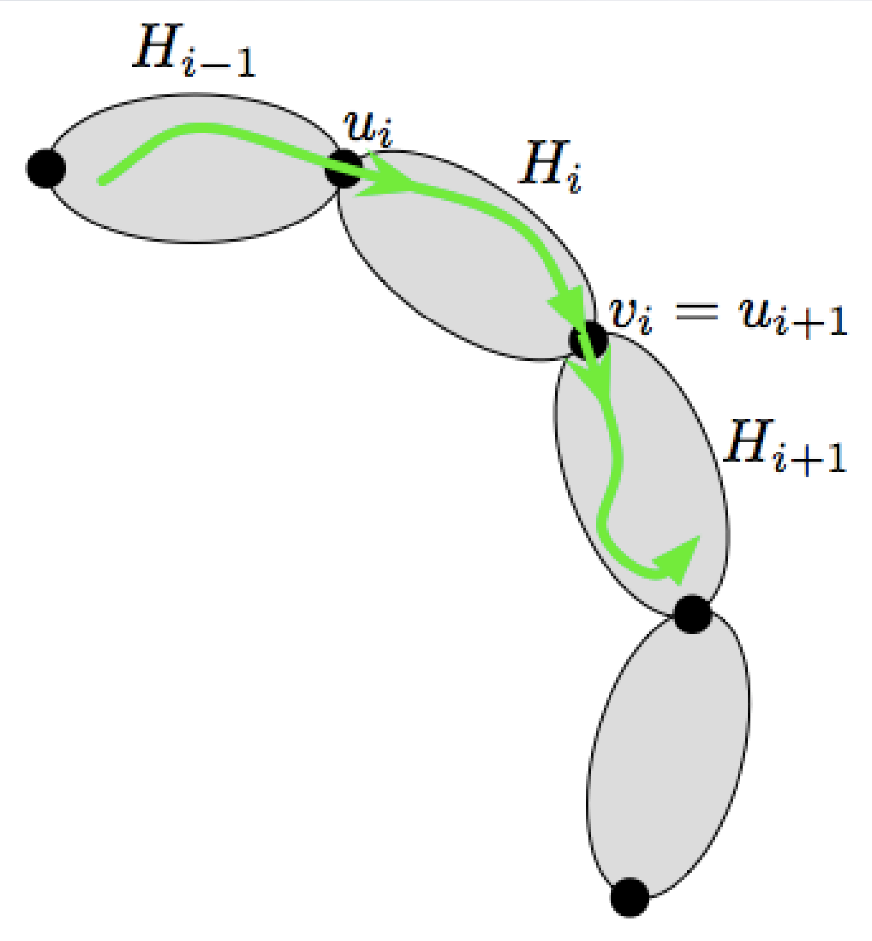

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- assume P uses an edge of H_i and both u_i and v_i.

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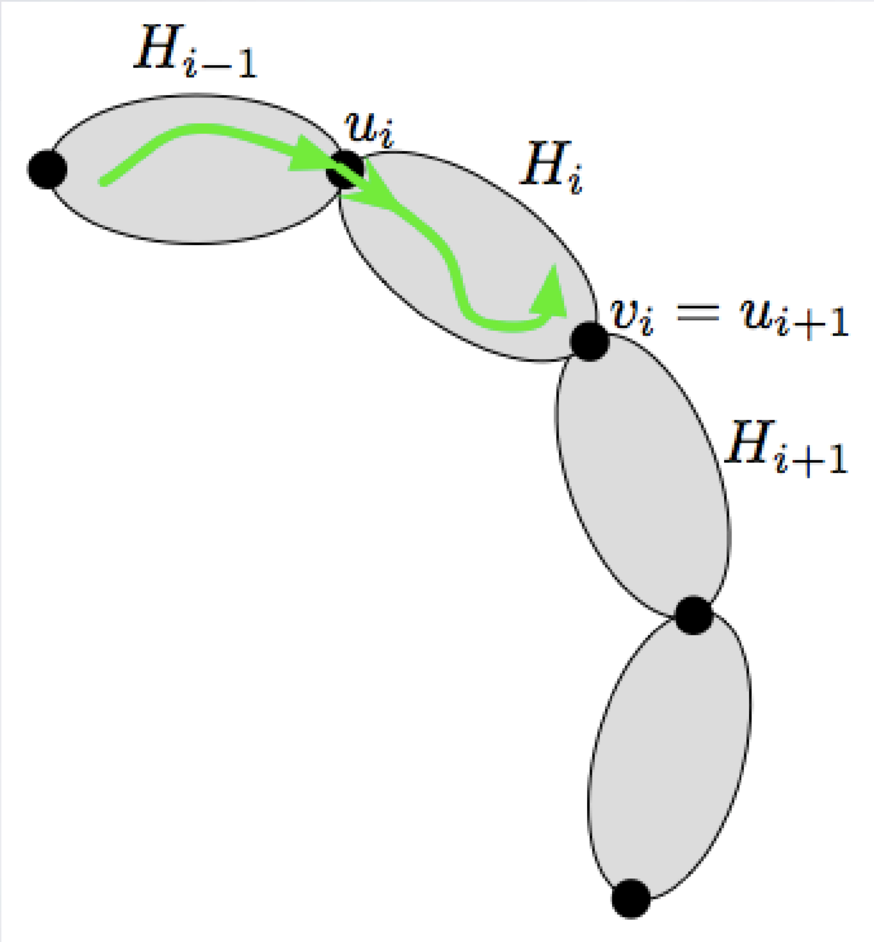

Let P be a k-alternating path, and pick P to intersect as few H_i as possible.

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- delete $P[H_i]$ and shift remaining path one bead to the right. Resulting path is still k-alternating

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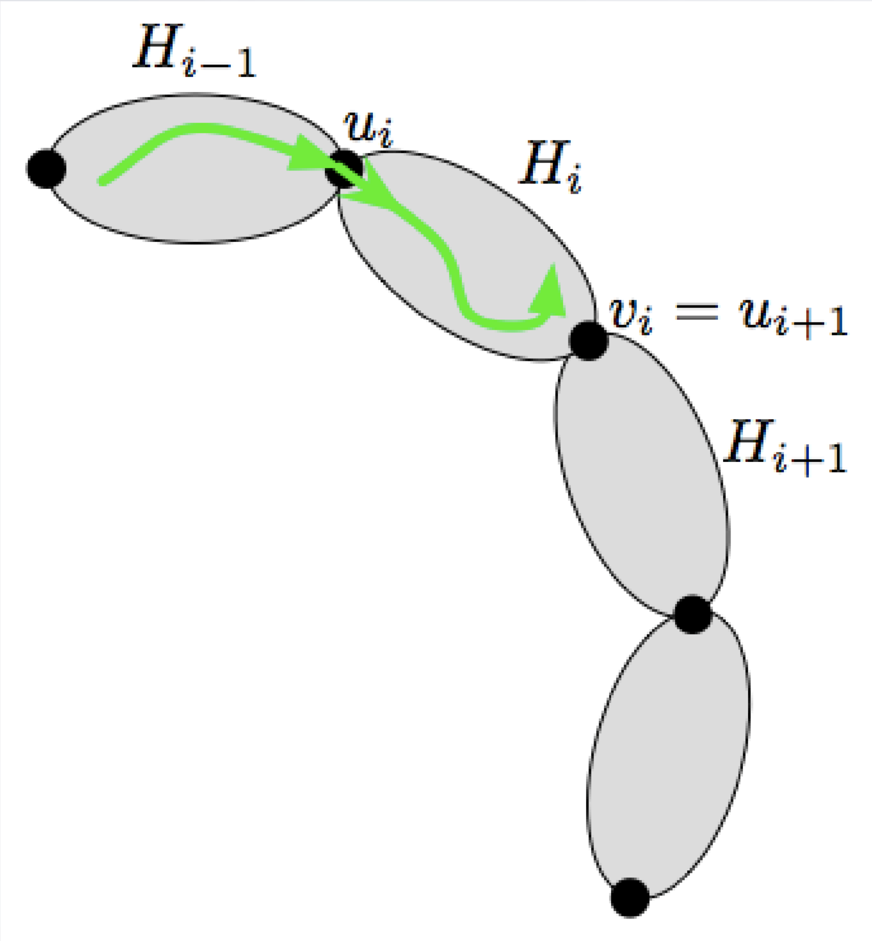


Let P be a k-alternating path, and pick P to intersect as few H_i as possible.

- assume P uses an edge of H_i and both u_i and v_i.

- delete $P[H_i]$ and shift remaining path one bead to the right. Resulting path is still k-alternating

Conclusion: no bounded set of vertices hits all maximal length alternating paths

Theorem: Let G be a digraph which doesn't contain a k-alternating path. If:

- un(G) is 2-connected, and
- there does not exist a separation (A, B) such that (G[B], $A \cap B$) is a series parallel triple,

then there exists f(k) vertices intersecting every (k-1)-alternating path.

Theorem: Series parallel triples (G, u, v) are well-quasi ordered under minors (while respecting the roots).

Proof of the theorem

Strengthen the induction hypothesis to consider digraphs with vertices AND edges labeled by a distinct wqo.

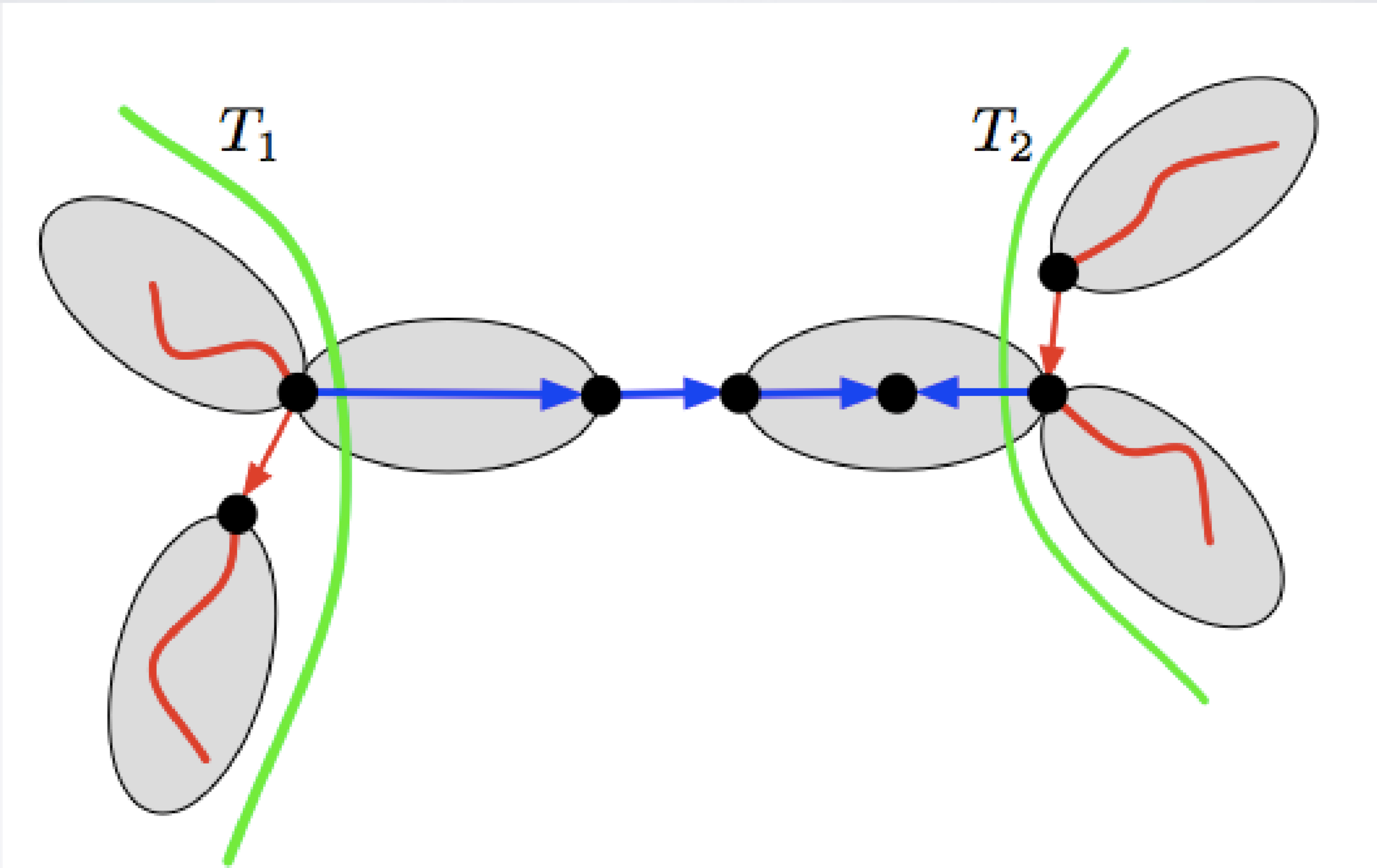
- For every series parallel two separation, replace the two cut with a labeled edge describing the deleted subgraph

- By induction, digraphs with no (k-1)-alternating path are wqo.

- 2-connected graphs with no k-alternating path are wqo by the packing result.

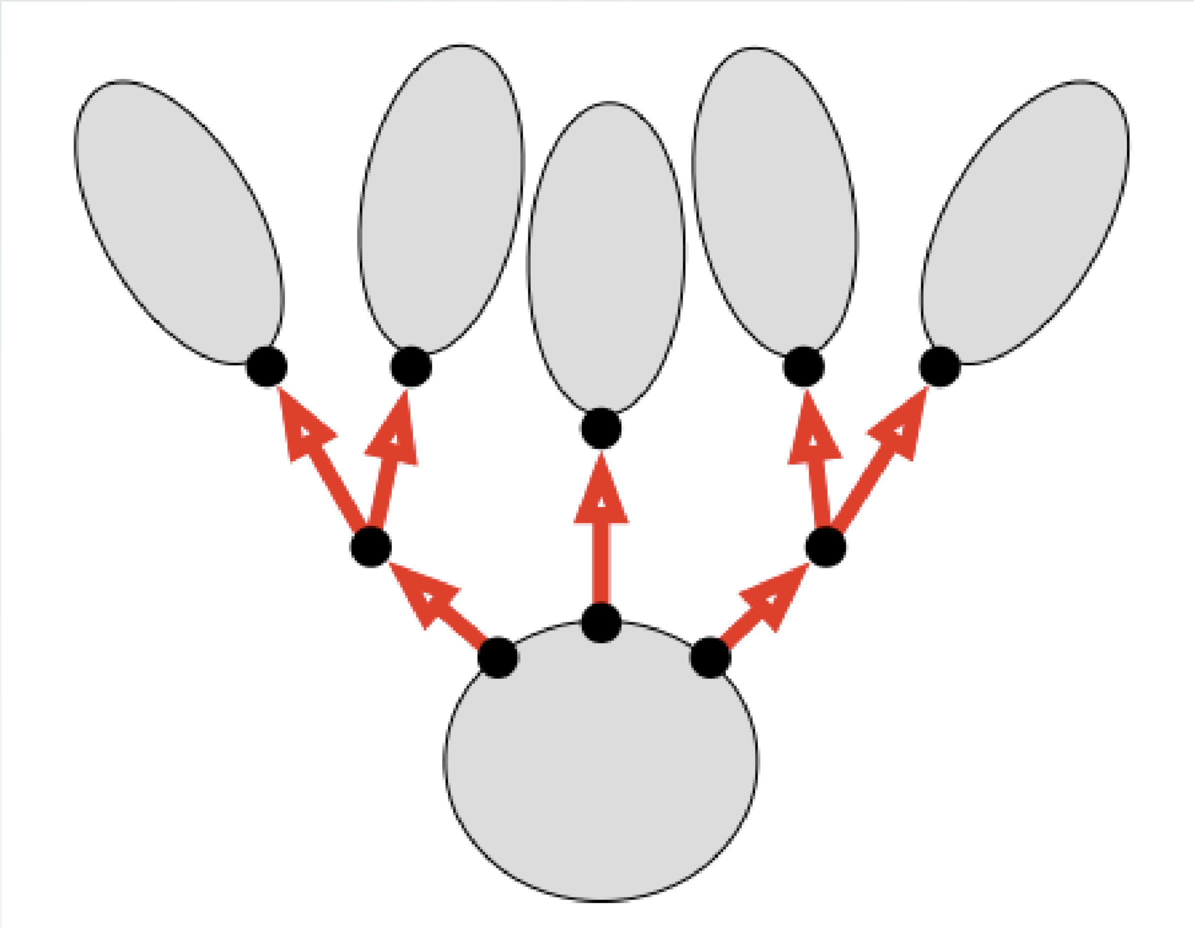
Proof of the theorem - Analyze the block decomposition

Consider two subtrees of the decomposition, each of which contains a (k-1)-alternating path.



Conclusion: All the blocks between them have a simple structure.

Proof of the theorem - Analyze the block decomposition



Structure like rooted arboresences which are (by induction) labeled with a wqo. Apply Nash-Williams' theorem on wqo of labeled trees.

Generalizations?

A (subraph/directed minor/topological minor) ideal **F** which excludes arbitrary long elements

- in a family of rooted paths
- in a family of cycle-like graphs

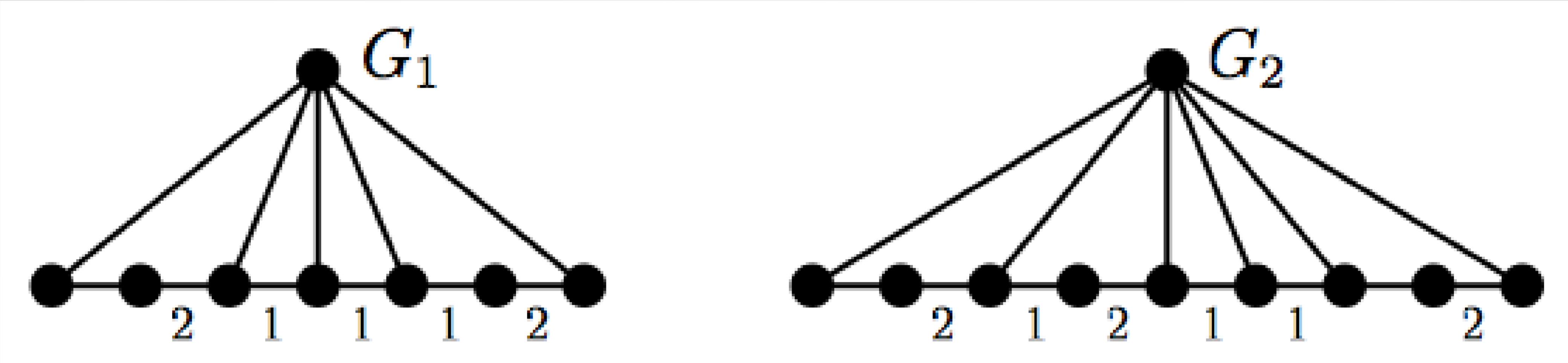
implies

- **F** is wqo under taking subgraphs (Ding)
- **F** is wqo under taking directed minors (CMOSW)
- **F** is wqo under taking topological minors (Liu Thomas)

3 theorems with all the intuitively the same canonical anti-chains.

Generalizations?

Induced subgraph have completely different anti-chains.



Associate to G_i a sequence S_i

➡ G_i is an induced subgraph of G_j if and only if S_i is an exact subsequence of S_j

Generalizations?

Question: Is there a general theorem for which these instances are special cases?